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General Formulas for Solving Solvable Sextic Equations

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Let G be a transitive, solvable subgroup of S_6. We show that there is a common formula for finding the roots of all irreducible sextic polynomials $f(x) \in \mathbf{Q}[x]$ with $\mathrm{Gal}(f) = G$. Moreover, once the roots r_i are calculated, there is an explicit procedure for numbering them so that the Galois group acts via $\tau(r_i) = r_{\tau(i)}$ for $\tau \in G \subset S_6$. We also demonstrate new criteria for determining the Galois group of an irreducible sextic polynomial $f(x) \in \mathbf{Q}[x]$. The first two results generalize results of D. S. Dummit (1991, *Math. Comp.* **57,** 387–401) for quintic polynomials.

Key Words: Galois theory; solvable polynomials; sextic polynomials.

1. INTRODUCTION

Given an irreducible polynomial $f(x) \in \mathbf{Q}[x]$, does there exist a formula for finding its roots using only the basic arithmetic operations and the taking of nth roots? To answer this classical question requires Galois theory, and the solution is one of the highpoints of an undergraduate course in aglebra. When such a formula exists, we say that the equation $f(x) = 0$ is solvable by radicals. If the same formula can be used for all polynomials $f(x)$ with degree n, we say that the general equation of degree n is solvable by radicals. The quadratic formula shows that the general equation of degree 2 is solvable by radicals. Similarly, the formulas

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of Cardano and Ferrari show that the general equations of degree 3 and 4 are solvable by radicals. For $n \geq 5$, Abel and Ruffini showed that the general equation of degree n is not solvable by radicals. Even stronger, Galois theory establishes that for each $n \geq 5$, there are irreducible polynomials $f(x) \in \mathbf{Q}[x]$ of degree n which are not solvable by radicals. In fact, for $n \geq 5$, most irreducible polynomials $f(x)$ of degree n are insolvable by radicals.

While there are no formulas involving radicals for finding the roots of each $f(x)$ of degree n when $n \geq 5$, we can ask whether such formulas exist when we restrict our attention to the class of polynomials which are solvable by radicals. To be precise, we recall that $f(x) = 0$ is solvable by radicals precisely when the Galois group $\mathrm{Gal}(f)$ of $f(x)$ is solvable. If we number the roots r_i of $f(x)$, then we can embed $\mathrm{Gal}(f)$ as a subgroup of S_n through its action upon the r_i. When $f(x)$ is irreducible, $\mathrm{Gal}(f)$ is a transitive subgroup of S_n. Since changing the numbering of the roots conjugates the embedding of $\mathrm{Gal}(f)$ in S_6, $\mathrm{Gal}(f)$ is not a well-defined function of f. Let $\overline{\mathrm{Gal}}(f)$ be the S_n-conjugacy class of $\mathrm{Gal}(f)$ and let Σ_n be the set of S_n-conjugacy classes of transitive subgroups of S_n. Then for each irreducible polynomial f, $\overline{\mathrm{Gal}}(f)$ gives a well-defined element of Σ_n. Given $G \in \Sigma_n$, we let Γ_G be the set of all irreducible polynomials $f(x) \in \mathbf{G}[x]$ with degree n such that $\overline{\mathrm{Gal}}\, f = G$.

Given $G \in \Sigma_n$, we say that the general equation of type (n, G) is *explicitly solvable* by radicals if

(i) There are formulas $z_1(t_i), z_2(t_i), \ldots, z_n(t_i)$ using only the basic arithmetic operations and radicals in variables $t_1, t_2, \ldots, t_m$

(ii) A number field K, $[K : \mathbf{Q}] < \infty$, and a bounded algorithm which associates to each $f \in \Gamma_G$, numbers $\hat{t}_1(f), \hat{t}_2(f), \ldots, \hat{t}_n(f) \in K$ such that $z_1(\hat{t}_i(f)), \ldots, z_n(\hat{t}_i(f))$ are the roots of f.

Traditionally, one would expect that one could take $K = \mathbf{Q}$. However, in cases such as the cubic or sextic, there are certain algebraic numbers independent of f which cannot be removed from the formulas. When $n = 3, 6$, we can take $K = \mathbf{Q}[\omega]$, where ω is a primitive cubic root of unity. We now pose the following conjecture:

Conjecture 1. Let n be a positive integer and $G \in \Sigma_n$ a conjugacy class of transitive, solvable subgroups of S_n. Then the general equation of type (n, G) is explicitly solvable by radicals.

The conjecture holds when $n \leq 4$ by the formulas mentioned above. Dummit [6] proved the conjecture when $n = 5$ and established explicit criteria for determining $\mathrm{Gal}(f)$. In this paper, we prove Conjecture 1 when $n = 6$. The set Σ_6 has 16 elements. To simplify the notation, we will let

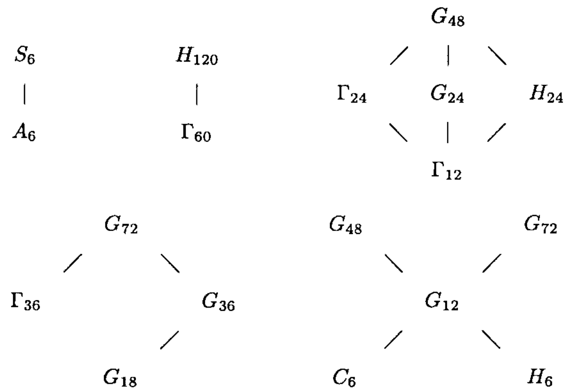

FIG. 1. Some subgroup relations between the transitive subgroups of S_6.

$\mathrm{Gal}(f)$ denote the element of $\overline{\mathrm{Gal}}(f)$ which appears in Fig. 1. It corresponds to an ordering of the roots of f.

THEOREM 1. *Let $G \in \Sigma_6$ be a conjugacy-class of transitive, solvable subgroups of S_6.*

(a) *The general equation of type $(6, G)$ is explicitly solvable by radicals.*

(b) *The formulas $z_i(t_j)$ and the algorithm for finding $\hat{t}_j(f)$ in (a) can be chosen so that for each $f \in \Gamma_G$, the Galois action of $\tau \in \mathrm{Gal}(f)$ on the roots $z_i = z_i(\hat{t}_j(f))$ of f is given by $\tau(z_i) = z_{\tau(i)}$.*

Let $f(x) \in \mathbf{Q}[x]$ be the irreducible sextic polynomial defined by

$$f(x) = x^6 - a_1x^5 + a_2x^4 - a_3x^3 + a_4x^2 - a_5x + a_6,$$

with roots r_i. We will prove Theorem 1 by determining explicit formulas for the roots r_i of $f(x)$. In the process, we will establish new criteria for determining $\mathrm{Gal}(f)$.

Before detailing our results, we would like to remark on a classical approach for proving Theorem 1. Let $K = \mathbf{Q}(x_1, \dots, x_6)$, where the x_i are indeterminates. There is a S_6-action on K defined by $\sigma(x_i) = x_{\sigma(i)}$, for $\sigma \in S_6$. Let $s_1 = \sum_i x_i$, $s_2 = \sum_{i<j} x_i x_j, \dots$ be the symmetric functions for the x_i and let $F = \mathbf{Q}(s_1, \dots, s_6)$. Let G be a transitive, solvable subgroup of S_6 and let K^G be the fixed field under G. Then $K^G = F(\theta)$ for some $\theta \in K$. The extension K/K^G is a solvable field extension with Galois group G and K is the splitting field for the polynomial

$$g(x) = x^6 - s_1x^5 + s_2x^4 - s_3x^3 + s_4x^2 - s_5x + s_6 \in F[x].$$

The x_i are the roots of $g(x)$ and thus there are formulas for x_i in terms of the s_j and θ involving only the basic arithmetic operations and radicals. It is natural to expect that one can then find formulas for the roots of $f(x)$ by substituting a_i for the s_i in these formulas and by determining the numerical value of θ when the r_i are substituted for the x_i. However, the difficulty is that the formulas for the x_i may involve rational functions in the s_i, θ. Hence, these functions may not be defined when specialized. Moreover, it may not always be possible to determine the value of θ when it is specialized. For "most" $f(x)$ these problems will not arise. While theoretically this approach could work for all $f(x)$, no one has yet been able to find the necessary θ (or finite set of θ). Instead, we will prove Theorem 1 in an alternate manner and will show that there is a common formula for handling all $f(x)$, including the exceptional cases.

We now describe our results in more detail. The set Σ_6 has 16 elements and the representatives of each conjugacy class are given in Fig. 1. Twelve of these groups are solvable, and there are two maximal solvable groups, G_{72} and G_{48}. When $\mathrm{Gal}(f)$ is solvable, we have

$$\mathrm{Gal}(f) \subset G_{72} \text{ or } \mathrm{Gal}(f) \subset G_{48}.$$

Let $b_i, c_i \in \mathbf{Q}$ be the rational numbers defined in the Appendix and let $f_{10}(x)$, $f_{15}(x) \in \mathbf{Q}[x]$ be the rational polynomials defined by

$$f_{10}(x) = x^{10} + \sum_{i=1}^{10} (-1)^i b_i x^{10-i}, f_{15}(x) = x^{15} + \sum_{i=1}^{15} (-1)^i c_i x^{15-i}.$$

We can use the rational roots for $f_{10}(x)$ and $f_{15}(x)$ to determine whether $\mathrm{Gal}(f)$ is solvable.

THEOREM 2. *If $f(x) \in \mathbf{Q}[x]$ is an irreducible sextic polynomial, then*

(a) $\mathrm{Gal}(f) \subset G_{72} \Leftrightarrow f_{10}(x)$ *has a rational root.*

(b) $\mathrm{Gal}(f) \subset G_{48} \Leftrightarrow$ *one of the following statements holds*:

(i) $f_{15}(x)$ *has a rational root with multiplicity* $\neq 3, 5$

(ii) $f_{15}(x)$ *has a rational root with multiplicity three and $f_{10}(x)$ has either an irreducible cubic factor or at least two distinct linear factors*

(iii) $f_{15}(x)$ *has a rational root with multiplicity five and $f_{10}(x)$ is reducible.*

As a corollary, we have

COROLLARY 1. *Let $f(x) \in \mathbf{Q}[x]$ be an irreducible, sextic polynomial. Then* $\mathrm{Gal}(f)$ *is solvable* $\Leftrightarrow$ *one of the following statements holds*:

(a) $f_{10}(x)$ *has a rational root*

(b) $f_{15}(x)$ *has a rational root with multiplicity* $\neq 5$

(c) $f_{15}(x)$ *has a rational root with multiplicity five and* $f_{10}(x)$ *is the product of irreducible quartic and sextic polynomials.*

If $\mathrm{Gal}(f)$ is not solvable, we can use the factorization of $f_{15}(x)$ to determine $\mathrm{Gal}(f)$ (see Proposition 5). Assume now that $\mathrm{Gal}(f)$ is solvable. Then exactly one of the following three cases must hold:

$$\begin{gathered}\text{(a)}\quad \mathrm{Gal}(f) \subset G_{72}, \mathrm{Gal}(f) \not\subset G_{48}, \\ \text{(b)}\quad \mathrm{Gal}(f) \subset G_{48}, \mathrm{Gal}(f) \not\subset G_{72}, \qquad \text{(c)}\quad \mathrm{Gal}(f) \subset G_{72} \cap G_{48}.\end{gathered} \tag{1}$$

In each case, there is a simple straightforward algorithm for explicitly determining $\mathrm{Gal}(f)$ and finding the roots of $f(x)$. Section 4 (resp. Sections 5, 6) describes the criteria for determining $\mathrm{Gal}(f)$ and formulas for finding the roots in case (a) (respectively, (b), (c)). Theorem 1 then results from combining Propositions 6, 9, and Theorem 6.

As an example, we present the criterion from Section 5 for determining the Galois group once it is known that $\mathrm{Gal}(f) \subset G_{48}$, $\mathrm{Gal}(f) \not\subset G_{72}$. In this case, we have $\mathrm{Gal}(f) = G_{48}, \Gamma_{24}, G_{24}, H_{24}$, or Γ_{12}. Let $\Delta = \prod_{i<j}(r_i - r_j)^2$ be the discriminant of $f(x)$. A formula for Δ in terms of the coefficients a_i of $f(x)$ is given in the Appendix. In Section 5, we introduce a number χ determined by $f(x)$. We then have:

THEOREM 3. *Let* $f(x) \in \mathbf{Q}[x]$ *be an irreducible sextic with* $\mathrm{Gal}(f) = G_{48}$, Γ_{24}, G_{24}, H_{24}, *or* Γ_{12}. *Then*

(a) $\mathrm{Gal}(f) = G_{48} \Leftrightarrow$ *none of the numbers* χ, Δ, $\chi\Delta$ *are squares in* $\mathbf{Q}$.

(b) $\mathrm{Gal}(f) = \Gamma_{24} \Leftrightarrow \Delta$ *is a square in* $\mathbf{Q}$, *but* χ, $\chi\Delta$ *are not squares in* $\mathbf{Q}$.

(c) $\mathrm{Gal}(f) = G_{24} \Leftrightarrow \chi$ *is a square in* $\mathbf{Q}$, *but* Δ, $\chi\Delta$ *are not squares in* $\mathbf{Q}$.

(d) $\mathrm{Gal}(f) = H_{24} \Leftrightarrow \chi\Delta$ *is a square in* $\mathbf{Q}$, *but* χ, Δ *are not squares in* $\mathbf{Q}$.

(e) $\mathrm{Gal}(f) = \Gamma_{12} \Leftrightarrow \chi$, Δ, *and* $\chi\Delta$ *are squares in* $\mathbf{Q}$.

Except for the need to factor $f_{10}(x)$ and $f_{15}(x)$ in special cases of Theorem 2, our criteria for determining $\mathrm{Gal}(f)$ and finding the roots of $f(x)$ will only require finding the rational roots of a small number of resolvent equations of degree 2, 10, and 15 and certain related equations. Our approach is essentially that which Dummit [6] used to answer the same questions in the case of a quintic polynomial. However, for the sextic, the computational difficulties are greater and new wrinkles to Dummit's method appear as there is more than one conjugacy class of maximal

solvable transitive subgroups of S_6. The two main differences are the following: First, we must distinguish which maximal transitive subgroup G_{72} or G_{48} contains $\mathrm{Gal}(f)$. Second, in the analogue of Corollary 1 for the quintic, Dummit is able to use a single sextic rational polynomial $g(x)$. Dummit can show that $g(x)$ has no repeated roots and can thus use Cramer's rule to find explicit formulas for the coefficients of the other polynomials needed in the paper. But in the case when $f(x)$ is a sextic polynomial, $f_{10}(x)$ and/or $f_{15}(x)$ will sometimes have repeated roots. Thus, if one uses Cramer's rule, one will obtain formulas that won't be defined for certain $f(x)$. Perhaps, by making different choices of $f_{10}(x)$, $f_{15}(x)$ one can obtain formulas that are always well defined, but this is not known. Instead in Sections 4–6, we find these coefficients by finding rational roots of other resolvent polynomials. In practical terms, this is quite simple and straightforward, but not as elegant. It would be interesting to see if it could be improved upon.

Previously, Girstmair [7] discovered criteria for determining the Galois group of a sextic using resolvents of degree 2, 6, and 10—though no formulas for the roots of $f(x)$ were derived. He showed that $\mathrm{Gal}(f)$ could be determined using a finite number of resolvents of these degrees. While in practice only a small number of resolvents are often needed, the number of resolvents needed could be quite high theoretically. Except when case (b)(ii) or (b)(iii) of Theorem 2 holds, our criteria improves Girstmair's criteria by showing that the rational roots of at most eight polynomials are needed to determine $\mathrm{Gal}(f)$. In the exceptional cases, one can either factor a rational polynomial of degree 10 or 15 or if one prefers to use only resolvents, use the set of resolvents constructed in [7]. Finally, we would also like to point out that there is a polynomial-time algorithm due to Landau and Miller [11] for determining whether an irreducible rational polynomial $f(x)$ of degree n is solvable (see [10] for an excellent overview). However, their point of view is quite different from our approach here. In particular, the Landau–Miller algorithm involves the factorization of polynomials over a general number field (even though $f(x)$ is rational) and is not concerned with the finding of "general" formulas for solving the equations.

We now describe the structure of this paper. After fixing notation, we explicitly describe the isomorphism classes of the 16 transitive subgroups of S_6 in Section 2. We then introduce Galois resolvents in Section 3 and study both their factorizations in the function field and the factorizations of their specializations $f_{10}(x)$, $f_{15}(x)$ in $\mathbf{Q}[x]$. Using this knowledge, we prove Theorem 2 and Corollary 1. In each case of (1), we determine other rational constants (e.g., χ, $\hat{M}$, $\hat{N}$) in Sections 4 and 5 and use them, as well as the discriminant Δ of $f(x)$, to precisely determine $\mathrm{Gal}(f)$. Each case in (1) requires a slightly different method and we establish the respective

criteria and formulas in Sections 4–6. Finally, we present two illustrative examples in Section 7 and discuss the computational aspects of the problem in Section 8.

2. TRANSITIVE SUBGROUPS OF S_6

In this section, we describe the transitive subgroups of S_6 (following [1], see also [2]). Cayley and Cole [4, 5] showed that each transitive subgroup of S_6 is conjugate in S_6 to one of sixteen non-isomorphic groups. We list these groups and some of the subgroup relations between them in Fig. 1. With the exception of S_6 and A_6, the subscript will denote the number of elements in the group. The groups $H_{m!}$ are isomorphic to S_m and the use of notation Γ_m in Fig. 1 indicates that $\Gamma_m = G_{2m} \cap A_6$—with the exception that $\Gamma_{60} = H_{120} \cap A_6$. One can also show that $\Gamma_{12} \cong A_4$ and $\Gamma_{60} \cong A_5$.

The four maximal transitive subgroups of S_6 are S_6, H_{120}, G_{72}, and G_{48}. We now explicitly describe their generators and subgroups. The group H_{120} is generated by the elements (1452), (16524), and (143562) and is isomorphic to S_5. $\Gamma_{60} = H_{120} \cap A_6$ is a subgroup of H_{120} of index 2 and is isomorphic to A_5.

We now consider G_{72} and its subgroups Γ_{36}, G_{36}, and G_{18}. Let $X = \{1,3,5\}$, $Y = \{2,4,6\}$, and let Sym_Z denote the symmetric group of a set Z. We regard Sym_X and Sym_Y as subgroups of S_6. Since $\sigma = (12)(34)(56) \in S_6$ acts on $\mathrm{Sym}_X \times \mathrm{Sym}_Y \subset S_6$ by conjugation, we can define the semi-direct product

$$G_{72} = (\mathrm{Sym}_X \times \mathrm{Sym}_Y) \rtimes \langle \sigma \rangle \subset S_6.$$

It is the stabilizer in S_6 of the set $S = \{X, Y\}$ and is generated by (13), (15), and σ. Now $\Gamma_{36} = A_6 \cap G_{72}$ is the subgroup of A_6 stabilizing S. The subgroup G_{36} is defined by $G_{36} = [A_6 \cap (\mathrm{Sym}_X \times \mathrm{Sym}_Y)] \rtimes \langle \sigma \rangle$. It is generated by (13)(24), (135), and σ. Finally, let G_{18} be the subgroup defined by $G_{18} = (A_X \times A_Y) \rtimes \langle \sigma \rangle$, where A_Z is the alternating subgroup of Sym_Z, for a set Z. The group G_{18} is generated by (135) and σ.

We now describe G_{48} and its transitive subgroups. Let $X = \{1,2\}$, $Y = \{3,4\}$, $Z = \{5,6\}$, and $T = \{X, Y, Z\}$. We define G_{48} to be the stabilizer of T in S_6. It is generated by the elements (12), (13)(24), and (135)(246). The subgroup $H = \mathrm{Sym}_X \times \mathrm{Sym}_Y \times \mathrm{Sym}_Z \cong (\mathbf{Z}/2\mathbf{Z})^3$ is generated by the cycles (12), (34), and (56). It is a normal subgroup of G_{48} of order 8 and $G_{48}/H \cong \mathrm{Sym}_T$. We have $G_{48} \cong \mathrm{Sym}_T \ltimes H$.

To define the subgroups of G_{48} we introduce two characters on G_{48}. Let $\alpha : G_{48} \to \{\pm 1\}$ be the restriction from S_6 to G_{48} of the sign homomorphism $S_6 \to \{\pm 1\}$. We let $\alpha_1 : G_{48} \to \{\pm 1\}$ be the composition of $G_{48} \to$

$G_{48}/H \cong \mathrm{Sym}_T$ and the sign homomorphism $\mathrm{Sym}_T \to \{\pm 1\}$. We now define the three groups Γ_{24}, G_{24}, and H_{24} to be the kernels in G_{48} of the respective homomorphisms α, α_1, and $\alpha\alpha_1$. We define $\Gamma_{12} = \Gamma_{24} \cap G_{24}$ $(= \Gamma_{24} \cap H_{24} = G_{24} \cap H_{24})$. In terms of generators, these groups are easily described. For example, G_{24} is generated by (12) and (135)(246), and Γ_{12} is generated by (135)(246) and (12)(34). We have $G_{24} \cong H_{24} \cong S_4$, but G_{24} and H_{24} are not conjugate groups in S_4.

Finally, we describe G_{12} and its two transitive subgroups C_6 and H_6 of order 6. We have $G_{12} = G_{72} \cap G_{48}$. It is generated by the cycles (135)(246), (13)(24), and (12)(34)(56). We denote by C_6 the cyclic subgroup generated by (145236) and by H_6, the group generated by (135)(246) and (12)(36)(45). We have $H_6 \cong S_3$.

3. GALOIS RESOLVENTS

Let $x_1, \dots, x_6$ be indeterminates over $\mathbf{Q}$, $R = \mathbf{Q}[x_1, \dots, x_6]$, and denote the quotient field of R by $K = \mathbf{Q}(x_1, \dots, x_6)$. We let $\sigma \in S_6$ act on K via $\sigma(x_i) = x_{\sigma(i)}$. Let $F = K^{S_6}$ be the field of elements of K fixed by S_6. Then $F = \mathbf{Q}(s_1, \dots, s_6)$, where

$$s_1 = x_1 + \cdots + x_6,$$
$$s_2 = \sum_{i<j} x_i x_j,$$
$$\vdots$$
$$s_6 = x_1 x_2 \cdots x_6,$$

are the symmetric polynomials in the x_i. K/F is a Galois extension with Galois group S_6. Given $\theta \in K$, we let $\mathrm{Stab}(\theta) = \{\sigma \in S_6 \mid \sigma(\theta) = \theta\}$. If $\theta \in K$ is a polynomial and $\mathrm{Stab}(\theta) = G$, we call θ a G-polynomial. If $G \subset S_6$, then K/K^G is a Galois extension with group G, and $K^G = F(\theta)$ for some $\theta \in K$ with $\mathrm{Stab}(\theta) = G$. Now θ will have $m = [S_6 : G]$ conjugates $\theta = \theta_1, \dots, \theta_m$ in K. The Galois resolvent of θ is defined as

$$F_\theta(x) = \prod_{i=1}^{m} (x - \theta_i) \in F[x].$$

$F_\theta(x)$ has degree m and is the product of distinct irreducible factors in $K^H[x]$ for each $H \subset S_6$. Let X be the set of left G-cosets in S_n. The group H acts on X by left multiplication. Elementary group theory shows that the degrees of the irreducible factors of $F_\theta(x)$ in $K^H[x]$ are given by the lengths of the H-orbits in X. Hence, the set of degrees of the

irreducible factors of $F_\theta(x)$ is independent of the choice of θ and depends only on G.

We now study three particular Galois resolvents. We denote by $F_2(x)$, $F_{10}(x)$, and $F_{15}(x)$ the Galois resolvents corresponding to the pairs (G, θ_G), for $G = A_6$, $\theta_{A_6} = \prod_{i<j}(x_i - x_j)$; $G = G_{72}$, $\theta_{72} = (x_1 + x_3 + x_5)(x_2 + x_4 + x_6)$; and $G = G_{48}$, $\theta_{48} = x_1x_2 + x_3x_4 + x_5x_6$. Whenever there is no ambiguity, we will often write F_d instead of F_G, where d will be the index $[S_6 : G]$. The degree of $F_d(x)$ is d. Table I indicates the degrees of the irreducible factors of these resolvents in $K^H[x]$, for all transitive subgroups $H \subset S_6$.

We now introduce the notion of specialization. Let $f(x) \in \mathbf{Q}[x]$ be an irreducible sextic polynomial. Choose a numbering $r_1, \dots, r_6$ of the roots of f so that the corresponding embedding $\mathrm{Gal}(f) \hookrightarrow S_6$ is one of the groups listed in Fig. 1. With this choice of r_i, the two definitions of $\mathrm{Gal}(f)$ defined in the Introduction agree. We note that there may be several ways to number the roots r_i so that this condition is fulfilled. We will need to distinguish between the action of S_6 on x_i and the action of $\mathrm{Gal}(f)$ on the roots r_i. Let L be the splitting field of $f(x)$ over $\mathbf{Q}$. Then let $\hat{\alpha} : R = \mathbf{Q}[x_1, \dots, x_6] \to L$ be the homomorphism defined by $\hat{\alpha}(x_i) = r_i$. Given $\theta \in R$, we let $\hat{\theta}$ denote the image $\hat{\alpha}(\theta) \in L$. If $g(x) = \sum_i a_i x^i \in R[x]$, we let $\hat{g}(x) = \sum_i \hat{a}_i x^i \in L[x]$. We will often use the following simple observation: If $\theta \in R$ is invariant under the action of $\mathrm{Gal}(f) \subset S_6$, then $\hat{\theta} \in \mathbf{Q}$.

TABLE I
The Degrees of the Irreducible Factors of the Galois Resolvents $F_2(x)$, $F_{10}(x)$, and $F_{15}(x)$ in $\mathbf{Q}(x_1, \dots, x_6)^G$

Group G	$F_2(x)$	$F_{10}(x)$	$F_{15}(x)$
S_6	2	10	15
A_6	1, 1	10	15
H_{120}	2	10	10, 5
Γ_{60}	1, 1	10	10, 5
G_{72}	2	9, 1	9, 6
Γ_{36}	1, 1	9, 1	9, 6
G_{36}	2	9, 1	9, 3, 3
G_{18}	2	9, 1	9, 3, 3
G_{48}	2	6, 4	8, 6, 1
Γ_{24}	1, 1	6, 4	8, 6, 1
G_{24}	2	6, 4	8, 6, 1
H_{24}	2	6, 4	6, 4, 4, 1
Γ_{12}	1, 1	6, 4	6, 4, 4, 1
G_{12}	2	6, 3, 1	6, 3, 3, 2, 1
C_6	2	6, 3, 1	6, 3, 3, 2, 1
H_6	2	3, 3, 3, 1	3, 3, 3, 3, 1, 1, 1

Similarly, if the coefficients of $g(x)$ are Gal(f)-invariant, then $\hat{g}(x) \in \mathbf{Q}[x]$. In particular, for each $G \subset S_6$ and G-polynomial $\theta \in R$, we have $\hat{F}_\theta(x) \in \mathbf{Q}[x]$.

It is important to remember that all specializations are with respect to $f(x)$ and a given numbering of the r_i. However, the specialization of a Galois resolvent can be computed without knowing r_i or their numbering. Let θ be a G-polynomial for some $G \subset S_6$ and $F_\theta(x)$ be its Galois resolvent. Then the coefficients of $\hat{F}_\theta(x)$ are polynomials in the coefficients a_i of $f(x)$. Hence $\hat{F}_\theta(x)$ can be computed by knowing only $f(x)$.

We now study several specific specializations of Galois resolvents. Again, for ease of notation, we will often write $f_G(x)$ or $f_d(x)$ (if $d = [S_6 : G]$) instead of $\hat{F}_G(x)$. For example, $f_2(x) = \hat{F}_2(x) = x^2 - \Delta$, where Δ is the discriminant of $f(x)$. Hence determining whether $f_2(x)$ has rational roots is equivalent to determining whether Gal$(f) \subset A_6$. We will use $f_{10}(x) = \hat{F}_{10}(x)$ and $f_{15}(x) = \hat{F}_{15}(x)$ to draw similar conclusions about Gal(f). The coefficients of $f_2(x)$, $f_{10}(x)$, and $f_{15}(x)$ are symmetric polynomials in the r_i and can be expressed as polynomials in the coefficients $a_1, \ldots, a_6$ of $f(x)$. Details of the computations are given in Section 8. In the Appendix, we list the formulas for Δ and the coefficients b_i, c_i of $f_{10}(x)$ and $f_{15}(x)$.

Let us now review how Galois resolvents can be used to calculate Gal(f). Let $G \subset S_6$ and choose a G-polynomial $\theta_G \in \mathbf{Q}[x_1, \ldots, x_6]$. Let $F_{\theta_G}(x)$ be the Galois resolvent. We will write Gal$(f) \subset_c G$ if Gal(f) is conjugate in S_6 to a subgroup of G. It can be easily shown:

PROPOSITION 1 [12]. *If* Gal$(f) \subset_c G$, *then* $\hat{F}_{\theta_G}(x) \in \mathbf{Q}[x]$ *has a rational root. Conversely, if* $\hat{F}_{\theta_G}(x)$ *has a rational root with multiplicity one, then* Gal$(f) \subset_c G$.

By assuming that Gal(f) is one of the 16 groups in Fig. 1, we can replace $\subset_c$ in Proposition 1 by $\subset$ when $G = G_{72}$, G_{48}, or A_6. If for each transitive subgroup $G \subset S_6$, there exists $\theta_G \in \mathbf{Q}[x_1, \ldots, x_6]$ such that the specialization $\hat{F}_{\theta_G}(x)$ always has distinct roots, then Proposition 1 would solve the problem of determining Galois groups. There has been some work done by Girstmair [8, 9] on whether such θ_G always exist, but this seems to be a difficult question and little is known.

The key to our approach is that we can choose θ_G for $G = G_{72}$, G_{48} so that $\hat{F}_{\theta_G}(x)$ has a rational root with multiplicity one most of the time. And in the remaining cases, we can use the factorization of $\hat{F}_{\theta_G}(x)$ to determine whether Gal$(f) \subset G$. Then, once we know whether Gal$(f) \subset G_{72}$ and Gal$(f) \subset G_{48}$, we can use other criteria to determine Gal(f) precisely. The three Galois resolvents we will use are $f_2(x) = x^2 - \Delta$, $f_{10}(x)$, and $f_{15}(x)$.

We now consider the factorization of $f_2(x)$, $f_{10}(x)$, $f_{15}(x)$ in $\mathbf{Q}[x]$ when Gal$(f) = G$, for each transitive subgroup G of S_6. The factorization of

$f_2(x)$ is easy to determine. $f_2(x) = x^2 - \Delta$ has a rational root if and only if $\mathrm{Gal}(f) \subset A_6$. And since $\Delta \neq 0$, these statements are equivalent to Δ being a square in $\mathbf{Q}$. For the other cases, we will make heavy use of the well-known lemma:

LEMMA 1. *Let $F_{\theta_G}(x)$ be the Galois resolvent associated to $G \subset S_6$. Assume that $f(x)$ is an irreducible polynomial with* $\mathrm{Gal}(f)$. *If $F(x)$ is an irreducible factor of the Galois resolvent $F_{\theta_G}(x)$ in $K^{\mathrm{Gal}(f)}[x]$, then $\hat{F}(x)$ is either an irreducible polynomial or the power of a linear polynomial in $\mathbf{Q}[x]$.*

We now consider the factorization of $f_{10}(x)$ in $\mathbf{Q}[x]$. We recall that $f_{10}(x) = \hat{F}_{10}(x)$. Let $\theta_{(abc)(def)} = (x_a + x_b + x_c)(x_d + x_e + x_f)$. Then the ten roots of $F_{10}(x)$ are the $\theta_{(abc)(def)}$, where $\{(abc)(def)\}$ ranges over all ten partitions of $\{1, \dots, 6\}$ into two sets of three elements each. Since $\theta_{(135)(246)} = \theta_{72}$ is a G_{72}-polynomial, if $\mathrm{Gal}(f) \subset G_{72}$, then $\theta = \hat{\theta}_{(135)(246)}$ is a rational root of $f_{10}(x)$.

PROPOSITION 2. *Let $f(x) \in \mathbf{Q}[x]$ be an irreducible sextic polynomial and assume* $\mathrm{Gal}(f)$ *is one of the groups in Fig.* 1.

(a) $\mathrm{Gal}(f) \subset G_{72} \Leftrightarrow f_{10}(x)$ *has a rational root. When this holds, $f_{10}(x)$ has a rational root with multiplicity one.*

(b) *If $F(x)$ is an irreducible factor of $F_{10}(x)$ in $K^{\mathrm{Gal}(f)}[x]$ of degree ≥ 4, then $\hat{F}(x)$ is an irreducible factor of $f_{10}(x)$ in $\mathbf{Q}[x]$.*

Proof. We first prove (b). By Table I, we can assume that $\mathrm{Gal}(f) \neq H_6$. Inspection then shows that whenever $F_{10}(x)$ has an irreducible factor $F(x)$ of degree 6, 9, or 10, then $F(x)$ has $\theta_{(123)(456)}$, $\theta_{(124)(356)}$, and $\theta_{(125)(346)}$ as roots. By Lemma 1, $\hat{F}(x)$ is either irreducible in $\mathbf{Q}[x]$ or

$$\hat{\theta}_{(123)(456)} = \hat{\theta}_{(124)(356)} = \hat{\theta}_{(125)(346)}.$$

Assume that the latter holds. Then the first equality shows that $r_1 + r_2 = r_5 + r_6$, the second gives $r_1 + r_2 = r_3 + r_6$, and we obtain the contradiction $r_3 = r_5$. Hence $\hat{F}(x) \in \mathbf{Q}[x]$ is irreducible. By process of elimination, we can now assume that $F(x)$ is the irreducible factor of degree four occurring when $\mathrm{Gal}(f) \subset G_{48}$, $\mathrm{Gal}(f) \not\subset G_{72}$. The roots of $F(x)$ are then $\theta_{(135)(246)}$, $\theta_{(136)(245)}$, $\theta_{(145)(236)}$, and $\theta_{(146)(235)}$. Again, by Lemma 1, if $\hat{F}(x)$ is not irreducible, then $\hat{\theta}_{(135)(246)} = \hat{\theta}_{(136)(245)}$ and $\hat{\theta}_{(145)(236)} = \hat{\theta}_{(146)(235)}$. Hence, $r_1 + r_3 = r_2 + r_4$ and $r_1 + r_4 = r_2 + r_3$ and we obtain the contradiction $r_1 = r_2$. Thus $\hat{F}(x)$ is irreducible and (b) is proved.

We now prove (a). The direction $(\Rightarrow)$ follows from Proposition 1. Now $(\Leftarrow)$ follows from (b) since if $\mathrm{Gal}(f) \not\subset G_{72}$, then $f_{10}(x)$ does not have a rational root. Hence the equivalence in (a) is proved. We now show that when it holds, $f_{10}(x)$ has a rational root with multiplicity one. When

$\mathrm{Gal}(f) \subset G_{72}$ $\mathrm{Gal}(f) \not\subset G_{48}$, then by (b), $f_{10}(x)$ has a rational root with multiplicity one. Hence we can assume that $\mathrm{Gal}(f) = G_{12}$, C_6, or H_6. We will show that in each case, $f_{10}(x)$ has a rational root with multiplicity one. We first consider the case when $\mathrm{Gal}(f) = H_6$. It suffices to show that if the specialization $\hat{F}(x)$ of an irreducible cubic factor $F(x)$ of $F_{10}(x)$ equals $(x-a)^3$, then $a \neq \theta$ $(= \hat{\theta}_{(135)(246)})$. Inspection shows that the roots of the three irreducible cubic factors are given by the sets

$$\left\{\hat{\theta}_{(136)(245)}, \hat{\theta}_{(145)(236)}, \hat{\theta}_{(146)(235)}\right\}, \qquad \left\{\hat{\theta}_{(132)(456)}, \hat{\theta}_{(126)(354)}, \hat{\theta}_{(156)(234)}\right\},$$
$$\left\{\hat{\theta}_{(134)(256)}, \hat{\theta}_{(146)(235)}, \hat{\theta}_{(145)(236)}\right\}.$$

If $a = \theta$, then either

$$\hat{\theta}_{(135)(246)} = \hat{\theta}_{(136)(245)} = \hat{\theta}_{(145)(236)} = \hat{\theta}_{(146)(235)},$$

or

$$\hat{\theta}_{(135)(246)} = \hat{\theta}_{(132)(456)} = \hat{\theta}_{(126)(354)} = \hat{\theta}_{(156)(234)},$$

or

$$\hat{\theta}_{(135)(246)} = \hat{\theta}_{(134)(256)} = \hat{\theta}_{(146)(235)} = \hat{\theta}_{(145)(236)}.$$

Proceeding as in the second part of the proof for (b), we obtain a contradiction in all three cases. The cases when $\mathrm{Gal}(f) = G_{12}$, C_6 are similar. Hence (a) is proved. ∎

More generally, for any Galois resolvent coming from a G_{72}-polynomial, we can show

PROPOSITION 3. *Let $f(x) \in \mathbf{Q}[x]$ be an irreducible sextic polynomial and assume that* $\mathrm{Gal}(f)$ *is one of the groups listed in Fig*. 1.

(a) *Let $\theta \in \mathbf{Q}[x_1, \dots, x_6]$ be a G_{72}-polynomial. If* $\mathrm{Gal}(f) \subset G_{72}$, *then $\hat{\theta} \in \mathbf{Q}$ and is the unique root of $\hat{F}_\theta(x) \in \mathbf{Q}[x]$ occurring with multiplicity* 1, 4, 7, *or* 10.

(b) *Let $\theta \in \mathbf{Q}[x_1, \dots, x_6]$ be a G_{48}-polynomial. If* $\mathrm{Gal}(f) \subset G_{48}$, $\mathrm{Gal}(f) \not\subset G_{72}$, *then $\hat{\theta} \in \mathbf{Q}$ and is the unique root of $\hat{F}_\theta(x) \in \mathbf{Q}[x]$ with multiplicity* 1, 5, 7, 9, 11, *or* 15.

Proof. We prove (a). Using Table I and Lemma 1, for each possible Galois group $\mathrm{Gal}(f) = G \subset G_{72}$, one can determine the possible decompositions of $\hat{F}_\theta(x)$ in $\mathbf{Q}[x]$. For each possible decomposition, inspection shows that there exists a positive integer n such that $\hat{\theta}$ is the unique root r

of $\hat{F}_\theta(x)$ with multiplicity n. The list of such n is the list in (a). Since no other root can have this multiplicity, (a) is proved. The proof of (b) is the same. The restriction $\mathrm{Gal}(f) \subset G_{48}$, $\mathrm{Gal}(f) \not\subset G_{72}$ occurs because when $\mathrm{Gal}(f) \subset G_{12}$, it can be the case that $\hat{\theta}$ cannot be determined because there are multiple roots with the same multiplicity. ∎

We now consider the factorization of $f_{15}(x)$ in $\mathbf{Q}[x]$. Let

$$\theta_{(ab)(cd)(ef)} = x_a x_b + x_c x_d + x_e x_f.$$

The roots of $F_{15}(x) = F_{\theta_{48}}(x)$ are the fifteen conjugates of θ_{48} listed in Fig. 2. We have

PROPOSITION 4. *Let $f(x) \in \mathbf{Q}[x]$ be an irreducible sextic polynomial. If $F(x)$ is an irreducible factor of $F_{15}(x)$ in $K^{\mathrm{Gal}(f)}[x]$ with degree $d \geq 6$, then $\hat{F}(x) \in \mathbf{Q}[x]$ is irreducible.*

Proof. Assume $\hat{F}(x)$ is reducible. Then $\hat{F}(x) = (x-a)^d$ for some $a \in \mathbf{Q}$, by Lemma 1. Since $\deg F(x) \geq 6$, two of the roots of $F(x)$ must be $\theta_{(1b)(cd)(ef)}$, $\theta_{(1b)(ce)(df)}$, for some permutation b, c, d, e, f of the numbers $\{2, \dots, 6\}$. Since $\hat{\theta}_{(1b)(cd)(ef)} = \hat{\theta}_{(1b)(ce)(df)}$, we have $(r_c - r_f)(r_d - r_e) = 0$ and thus, either the contradiction $r_c = r_f$ or the contradiction $r_d = r_e$. Hence $\hat{F}(x)$ is irreducible. ∎

We now prove Theorem 2.

Proof of Theorem 2. Part (a) is proved in Proposition 2(a). We now prove (b)($\Rightarrow$). Assume $\mathrm{Gal}(f) \subset G_{48}$. By Proposition 1, $f_{15}(x)$ has a rational root. If $f_{15}(x)$ has a rational root with multiplicity $\neq 3, 5$, then condition (i) holds. We now assume that (i) does not hold. If $f_{15}(x)$ has a rational root with multiplicity 3, then by Proposition 4 and Table I, either $\mathrm{Gal}(f) = G_{12}$, C_6, or H_6. Inspection, along the lines of the proof of Proposition 2(b), then shows that the criterion in condition (ii) holds. If $\mathrm{Gal}(f) \subset G_{48}$ and $f_{15}(x)$ does not have any rational roots except with multiplicity 5, then by Proposition 4 we have $\mathrm{Gal}(f) = H_{24}$ or Γ_{12}. Then

$\theta_{(12)(34)(56)}$	$\theta_{(12)(35)(46)}$	$\theta_{(12)(36)(46)}$
$\theta_{(13)(24)(56)}$	$\theta_{(13)(25)(46)}$	$\theta_{(13)(26)(45)}$
$\theta_{(14)(23)(56)}$	$\theta_{(14)(25)(36)}$	$\theta_{(14)(26)(35)}$
$\theta_{(15)(23)(46)}$	$\theta_{(15)(24)(36)}$	$\theta_{(15)(26)(34)}$
$\theta_{(16)(23)(45)}$	$\theta_{(16)(24)(35)}$	$\theta_{(16)(25)(34)}$

FIG. 2. Roots of $F_{15}(x)$.

since $F_{10}(x)$ is reducible in $K^G[x]$ when $G = H_{24}$ or Γ_{12}, $f_{10}(x)$ is reducible. Hence (iii) holds.

We now prove (b)($\Leftarrow$). Assume that condition (i) holds. Then by Proposition 4, we must have $\mathrm{Gal}(f) \subset G_{48}$. Assume now that condition (ii) holds. Since there is a root with multiplicity 3, then by Table I and Proposition 4, we must have $\mathrm{Gal}(f) = G_{36}$, G_{18} or $\mathrm{Gal}(f) \subset G_{12}$. But Proposition 2 shows that $f_{10}(x)$ contains an irreducible factor of degree 9 when $\mathrm{Gal}(f) = G_{36}$, G_{18}. Hence $\mathrm{Gal}(f) \subset G_{12} \subset G_{48}$. Finally, assume that condition (iii) holds. If $f_{15}(x)$ has a root with multiplicity 5 then $\mathrm{Gal}(f) = H_{120}$, Γ_{60}, or $\mathrm{Gal}(f) \subset G_{48}$ by Lemma 1. Since $f_{10}(x)$ is reducible, by Proposition 2 we have $\mathrm{Gal}(f) \neq H_{120}$, Γ_{60}. Hence $\mathrm{Gal}(f) \subset G_{48}$ and the theorem is proven. ∎

We now prove Corollary 1.

Proof of Corollary 1. $\mathrm{Gal}(f)$ is solvable if and only if $\mathrm{Gal}(f) \subset G_{72}$ or $\mathrm{Gal}(f) \subset G_{48}$. Hence Corollary 1 follows from Theorem 2, Table I, and the observation that $f_{15}(x)$ can only have a rational root with multiplicity three when $\mathrm{Gal}(f) \subset G_{72}$ or $\mathrm{Gal}(f) \subset G_{48}$. ∎

Once it is known by Theorem 2 that $\mathrm{Gal}(f)$ is not solvable, it is easy to determine $\mathrm{Gal}(f)$. We have

PROPOSITION 5. *Let $f(x) \in \mathbf{Q}[x]$ be a non-solvable irreducible, sextic polynomial. Then*

(a) $\mathrm{Gal}(f) \cong S_6 \Leftrightarrow f_{15}(x)$ *is irreducible in* $\mathbf{Q}[x]$ *and* Δ *is not a square in* $\mathbf{Q}$.

(b) $\mathrm{Gal}(f) \cong A_6 \Leftrightarrow f_{15}(x)$ *is irreducible in* $\mathbf{Q}[x]$ *and* Δ *is a square in* $\mathbf{Q}$.

(c) $\mathrm{Gal}(f) \cong H_{120} \Leftrightarrow f_{15}(x)$ *is reducible in* $\mathbf{Q}[x]$ *and* Δ *is not a square in* $\mathbf{Q}$.

(d) $\mathrm{Gal}(f) \cong \Gamma_{60} \Leftrightarrow f_{15}(x)$ *is reducible in* $\mathbf{Q}[x]$ *and* Δ *is a square in* $\mathbf{Q}$.

Proof. We have $\mathrm{Gal}(f) \cong S_6$, A_6, H_{120}, or Γ_{60}. By Proposition 4, $f_{15}(x)$ is irreducible $\Leftrightarrow$ $\mathrm{Gal}(f) \cong S_6$ or A_6. The discriminant Δ distinguishes the remaining cases. ∎

4. SOLVING THE SEXTIC: $\mathrm{Gal}(f) \subset G_{72}, \mathrm{Gal}(f) \not\subset G_{48}$

As we proved Theorem 2 in the previous section, we are now able to distinguish the three cases in (1). In this section, we assume that $\mathrm{Gal}(f) \subset G_{72}$ and $\mathrm{Gal}(f) \not\subset G_{48}$. We describe how to determine the roots of the sextic $f(x)$, determine $\mathrm{Gal}(f)$, and explicitly determine the action of

$\mathrm{Gal}(f)$ on the roots once they are found. We first explain how to determine $\mathrm{Gal}(f)$. Let

$$\beta_1 = (x_1 - x_3)(x_3 - x_5)(x_5 - x_1),$$
$$\beta_2 = (x_2 - x_4)(x_4 - x_6)(x_6 - x_2), \qquad \delta = \beta_1 + \beta_2, \mu = \beta_1 - \beta_2, \tag{2}$$
$$\text{and} \qquad M = \delta^2 + \mu^2, N = \delta^2\mu^2.$$

Now M and N are G_{72}-polynomials. Let $f_M(x)$, $f_N(x)$ denote the specializations of their Galois resolvents. $\hat{M}$, $\hat{N}$ are rational roots of $f_M(x)$, $f_N(x)$, respectively, which can be determined by Proposition 3. Let

$$g(x) = x^2 - \hat{M}x + \hat{N} \in \mathbf{Q}[x], \tag{3}$$

and recall that Δ is the discriminant of $f(x)$ defined in the Appendix.

THEOREM 4. *Suppose $f(x) \in \mathbf{Q}[x]$ is an irreducible sextic with* $\mathrm{Gal}(f) = G_{72}, \Gamma_{36}, G_{36}$, *or* G_{18}. *Then*

(a) $\mathrm{Gal}(f) = \Gamma_{36} \Leftrightarrow \Delta$ *is a square in* $\mathbf{Q}$.

(b) $\mathrm{Gal}(f) = G_{72} \Leftrightarrow \Delta$ *is not a square in* $\mathbf{Q}$ *and* $g(x) \in \mathbf{Q}[x]$ *is irreducible*.

(c) $\mathrm{Gal}(f) \subset G_{36} \Leftrightarrow \Delta$ *is not a square in* $\mathbf{Q}$ *and* $g(x) \in \mathbf{Q}[x]$ *is reducible*.

Proof. We first prove (a). Of the four groups, only Γ_{36} is a subset of A_6. Hence Δ is a rational square $\Leftrightarrow \mathrm{Gal}(f) \subset A_6 \cap G_{72} \Leftrightarrow \mathrm{Gal}(f) = \Gamma_{36}$ and (a) is proved. We now prove (b). By (a), we can assume that Δ is not a rational square and $\mathrm{Gal}(f) = G_{72}, G_{36}$, or G_{18}. Since $\beta_1, \beta_2 \neq 0$, we have $\delta^2 \neq \mu^2$. Since δ^2, μ^2 are G_{36}-polynomials which are permuted by the action of $(24) \in G_{72}$, we have $g(x)$ is irreducible if and only if $(24) \in \mathrm{Gal}(f)$. Hence (b) is proved and (c) follows immediately. ∎

We cannot use Theorem 4 to distinguish between the cases $\mathrm{Gal}(f) = G_{36}$ and $\mathrm{Gal}(f) = G_{18}$. We now establish criteria to determine $\mathrm{Gal}(f)$ when $\mathrm{Gal}(f) \subset G_{36}$. We assume $\mathrm{Gal}(f) \subset G_{36}$ and introduce some necessary notation. Let

$$\begin{aligned} &d_1 = x_1 + x_3 + x_5, \qquad d_1' = x_2 + x_4 + x_6,\\ &d_2 = x_1x_3 + x_3x_5 + x_1x_5, \qquad d_2' = x_2x_4 + x_4x_6 + x_2x_6,\\ &d_3 = x_1x_3x_5, \qquad d_3' = x_2x_4x_6,\\ &\alpha_i = d_i - d_i', \qquad \gamma_i = d_i + d_i', \qquad \text{and} \qquad \delta_{ij} = \alpha_i\alpha_j, \end{aligned} \tag{4}$$

for $1 \leq i, j, \leq 3$. With the exception of the S_6-polynomial $\gamma_1 = s_1$, the α_i^2

and γ_i are G_{72}-polynomials. By Proposition 3, we can determine $\hat{\alpha}_i^2$ and $\hat{\gamma}_i$ as the rational roots of their respective Galois resolvents.

LEMMA 2. *Let $f(x) \in \mathbf{Q}[x]$ be an irreducible sextic polynomial with* $\mathrm{Gal}(f) \subset G_{36}$. *Then $\hat{\alpha}_i \neq 0$ for some i.*

Proof. Suppose that $\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\alpha}_3 = 0$. Then $\hat{d}_i = \hat{d}'_i$ for all i and $\hat{d}_i = \hat{\gamma}_i/2 \in \mathbf{Q}$. Hence, r_1, r_3, and r_5 are roots of the rational polynomial $2x^3 - \hat{\gamma}_1 x^2 + \hat{\gamma}_2 x - \hat{\gamma}_3$. Since this contradicts the irreducibility of $f(x)$, at least one of the $\hat{\alpha}_i$ is non-zero. ∎

Using the lemma, we fix a choice of j such that $\hat{\alpha}_j \neq 0$ and define

$$\begin{aligned} \delta_1 &= d_j \beta_1 + d'_j \beta_2, \qquad \mu_1 = d_j \beta_1 - d'_j \beta_2, \\ M_1 &= \delta_1^2 + \mu_1^2, \qquad N_1 = \delta_1^2 \mu_1^2. \end{aligned} \tag{5}$$

M_1, N_1 are G_{72}-polynomials and $\hat{M}_1$, $\hat{N}_1$ are values which can be effectively determined as rational roots of the Galois resolvents $f_{M_1}(x)$, $f_{N_1}(x)$. We then define

$$g_1(x) = x^2 - \hat{M}_1 x + \hat{N}_1 \in \mathbf{Q}[x]. \tag{6}$$

The roots of $g_1(x)$ are $\hat{\delta}_1^2$, $\hat{\mu}_1^2$. Since δ_1^2, μ_1^2 are G_{36}-polynomials and we are now assuming $\mathrm{Gal}(f) \subset G_{36}$, we have $\hat{\delta}_1^2$, $\hat{\mu}_1^2 \in \mathbf{Q}$ and $g_1(x)$ is a reducible polynomial. The following lemma shows that at least two of the numbers $\hat{\delta}$, $\hat{\mu}$, $\hat{\delta}_1$, and $\hat{\mu}_1$ are non-zero.

LEMMA 3. *Let $f(x) \in \mathbf{Q}[x]$ be an irreducible sextic polynomial with* $\mathrm{Gal}(f) \subset G_{36}$. *Then either $\hat{\delta}\hat{\mu} \neq 0$ or $\hat{\delta}_1 \hat{\mu} \neq 0$ or $\hat{\delta}\hat{\mu}_1 \neq 0$.*

Proof. We first note that if $\hat{\delta}_1 = 0$ and $\hat{\mu}_1 = 0$, then $\hat{d}_j \hat{\beta}_1 = \hat{d}'_j \hat{\beta}_2 = 0$. Since $\hat{\beta}_1$, $\hat{\beta}_2 \neq 0$, we have $\hat{d}_j = \hat{d}'_j = 0$ and $\hat{\alpha}_j = 0$. Since this contradicts the choice of j, one of the $\hat{\delta}_1$, $\hat{\mu}_1$ must be zero. A similar argument shows that either δ or μ must be non-zero. The lemma will then follow once we show that we cannot have $\hat{\delta} = \hat{\delta}_1 = 0$ or $\hat{\mu} = \hat{\mu}_1 = 0$. Assume that $\hat{\delta} = \hat{\delta}_1 = 0$. Then $\hat{\beta}_1 = -\hat{\beta}_2$ and $\hat{d}_j \hat{\beta}_1 = -\hat{d}'_j \hat{\beta}_2$. Hence $\hat{d}_j = \hat{d}'_j$, which contradicts the choice of j. Similarly, the assumption $\hat{\mu} = \hat{\mu}_1 = 0$ leads to a contradiction. ∎

The following theorem establishes a criterion for distinguishing the case $\mathrm{Gal}(f) = G_{36}$ from the case $\mathrm{Gal}(f) = G_{18}$.

THEOREM 5. *Suppose $f(x) \in \mathbf{Q}[x]$ is an irreducible sextic with* $\mathrm{Gal}(f) =$ G_{36} *or* G_{18}. *Then*

(a) *If both roots of* $g(x)$ *are non-zero, then* $\mathrm{Gal}(f) = G_{36} \Leftrightarrow$ *neither of the roots of* $g(x)$ *are squares in* $\mathbf{Q}$. *If* 0 *is a root of* $g(x)$, *then* $\mathrm{Gal}(f) = G_{36} \Leftrightarrow$ *none of the non-zero roots of* $g(x)$, $g_1(x)$ *are squares in* $\mathbf{Q}$.

(b) $\mathrm{Gal}(f) = G_{18} \Leftrightarrow$ *at least one of the non-zero roots of* $g(x)$, $g_1(x)$ *is a square in* $\mathbf{Q}$.

Proof. We first note that it suffices to prove (a). Let $\sigma \in G_{18}$ denote the element (12)(34)(56). Since $\sigma(\mu) = -\mu$ and $\sigma(\mu_1) = -\mu_1$, we see that $\hat{\mu}^2$, $\hat{\mu}_1^2$ cannot be non-zero rational squares. Now δ and δ_1 are G_{18}-polynomials. The element $\tau = (13)(24) \in G_{36} - G_{18}$ acts via $\tau(\delta) = -\delta$, $\tau(\delta_1) = -\delta_1$. Hence, if $\hat{\delta}$ (resp. $\hat{\delta}_1$) is non-zero, then $\hat{\delta}$ (resp. $\hat{\delta}_1$) is a square in $\mathbf{Q}$ if and only if $\tau \notin \mathrm{Gal}(f)$. By Lemma 3, either $\hat{\delta}$ or $\hat{\delta}_1$ is non-zero. Both assertions in (a) then follow. ∎

We now turn our attention to finding the roots of $f(x)$. Despite the order of our presentation, the formulas for finding the roots of $f(x)$ do not depend upon precisely knowing $\mathrm{Gal}(f)$.

PROPOSITION 6. *Let* $G \subset G_{72}$ *be one of the groups in Fig.* 1. *Let* $\overline{G} \in \Sigma_6$ *be the conjugacy class containing* G.

(a) *The general equation of type* $(6, \overline{G})$ *is explicitly solvable by radicals.*

(b) *If* $G_{18} \subset G \subset G_{72}$, *then the formulas* $z_i(t_j)$ *and the algorithm in* (a) *can be chosen so that for each* $f \in \Gamma_{\overline{G}}$, *the Galois action of* $\tau \in \mathrm{Gal}(f)$ *on the roots* $z_i = z_i(\hat{t}_j(f))$ *is given by* $\tau(z_i) = z_{\tau(i)}$.

Proof. (a) Let $f \in \Gamma_{\overline{G}}$ be an irreducible sextic polynomial with $\mathrm{Gal}(f) = G$. We recall that the specialization of the resolvents with respect to f depends upon numbering the roots r_i of f so that $\mathrm{Gal}(f) = G$. We will prove the proposition by finding a formula for the r_i. Define the G_{72}-polynomial δ_{ij} as in (4). Then $\hat{\delta}_{ij} \in \mathbf{Q}$ can be determined as a root of its resolvent $f_{\delta_{ij}}(x)$. By Lemma 2, we fix j to be the smallest j such that $\hat{\alpha}_j^2 = \hat{\delta}_{jj} \neq 0$. Let $v = \hat{\alpha}_j^2$ and let $v_j = \hat{\delta}_{ij}/\sqrt{v}$. There exists $\varepsilon_1 = \pm 1$ such that $v_j = \varepsilon_1 \hat{\alpha}_j$ for all j. Now for $\varepsilon = \pm 1$, define $e_i(\varepsilon) = \varepsilon v_i / 2\sqrt{v} + (1/2)\hat{\gamma}_i$. Then $e_i(\varepsilon_1)$ are the ith symmetric functions for r_1, r_3, r_5, and $e_i(-\varepsilon_1)$ are the ith symmetric functions for r_2, r_4, r_6. Let f_ε be the cubic polynomial $f_\varepsilon = x^3 - e_1(\varepsilon)x^2 + e_2(\varepsilon)x - e_3(\varepsilon)$. Define for $i = 1, 2, 3$, $z(\varepsilon, i) = y_{f_\varepsilon}(\omega^i)$, where ω is a primitive cubic root of unity. It is a formula in the variables ε, v, v_i, γ_i that uses only the basic arithmetic operations and radicals. Let $z_{2i-1} = z(1, i)$, $z_{2i} = z(-1, i)$ for $i = 1, 2, 3$. The z_1, z_3, z_5 are the roots of $f_1(x)$ and the z_2, z_4, z_6 are the roots of $f_{-1}(x)$. Since r_1, r_3, r_5 are the roots of $f_{\varepsilon_1}(x)$, we have

$$\{\{r_1, r_3, r_5\}, \{r_2, r_4, r_6\}\} = \{\{z_1, z_3, z_5\}, \{z_2, z_4, z_6\}\}. \tag{7}$$

Formulas when $\mathrm{Gal}(f) \subset G_{72}$.

1. Let $f(x) = x^6 - a_1x^5 + a_2x^4 - a_3x^3 + a_4x^2 - a_5x + a_6$.
2. Use Theorem 2 to verify $\mathrm{Gal}(f) \subset G_{72}$.
3. Define $z = \gamma_i, \delta_{ij}$ as in (4) and calculate the G_{72}-resolvent $F_z(x)$ for $z = \delta_{ij}$, $1 \leq i \leq j \leq 3$, γ_2, and γ_3.
4. For each z in 2., determine the value of $\hat{z}$ as the unique rational root of $\hat{F}_z(x)$ with multiplicity 1, 4, 7, or 10.
5. Choose j such that $\hat{\delta}_{jj} \neq 0$.
6. Let $v_i = \hat{\delta}_{ij}$, $v = v_j$. For $\varepsilon = \pm 1$, define $e_i(\varepsilon) = \dfrac{\varepsilon v_i}{2\sqrt{v}} + \dfrac{1}{2}\hat{\gamma}_i$. Let f_ε be the cubic polynomial $f_\varepsilon = x^3 - e_1(\varepsilon)x^2 + e_2(\varepsilon)x - e_3(\varepsilon)$.
7. Let $z(\varepsilon, i) = y_{f_\varepsilon}(\omega^i)$, where the y_g are the formulas defined in Figure 6 for the roots of a cubic polynomial g and ω is a primitive cubic root of unity.
8. $z(\varepsilon, i)$ is a formula using only the basic arithmetic operations and radicals in the variables ε, v, v_i, $\hat{\gamma}_i$.
9. Let $z_{2i-1} = z(1, i)$, $z_{2i} = z(-1, i)$ for $i = 1, 2, 3$. The z_i are the roots of $f(x)$.
10. If $\mathrm{Gal}(f) \not\subset G_{48}$, the Galois action of $\tau \in \mathrm{Gal}(f)$ on the z_i is given by $\tau(z_i) = z_{\tau(i)}$.

FIG. 3. Formulas for finding the roots of an irreducible, sextic $f(x) \in \mathbf{Q}[x]$ when $\mathrm{Gal}(f) \subset G_{72}$.

Hence there are formulas for finding the roots of f. Since there is a finite algorithm for calculating v, v_i, $\hat{\gamma}_i$, whose values all lie in $K = \mathbf{Q}$, we have proved (a).

(b) It suffices to show that for any set of roots z_i arising from the formulas in (a), that there is an automorphism $\alpha : \mathrm{Gal}(f) \to \mathrm{Gal}(f)$ satisfying $\alpha(\sigma)(z_i) = z_{\sigma(i)}$, for $\sigma \in \mathrm{Gal}(f)$. Then, by twisting the Galois action by α, (b) holds. Now the roots r_i were initially chosen so that $\mathrm{Gal}(f)$ was one of the four subgroups G_{72}, G_{36}, Γ_{36}, G_{18} and have the property that $\sigma(r_i) = r_{\sigma(i)}$ for $\sigma \in \mathrm{Gal}(f)$. The proof of (a) shows that the z_i can be numbered such that (7) holds. Hence there exists $\tau \in G_{72}$ such that $z_i = r_{\tau(i)}$. We then have $\sigma(z_i) = z_{\tau^{-1}\sigma\tau(i)}$. Since τ normalizes each of the groups $G_{72}, G_{36}, \Gamma_{36}, G_{18}$, we have $\tau\sigma\tau^{-1} \in \mathrm{Gal}(f)$ and $(\tau\sigma\tau^{-1})(z_i) = z_{\sigma(i)}$. Hence, letting $\alpha(\sigma) = \tau\sigma\tau^{-1}$ gives the desired map. ∎

5. SOLVING THE SEXTIC: $\mathrm{Gal}(f) \subset G_{48}, \mathrm{Gal}(f) \not\subset G_{72}$

In this section, we assume that $\mathrm{Gal}(f) \subset G_{48}$ and $\mathrm{Gal}(f) \not\subset G_{72}$. We now describe how to determine $\mathrm{Gal}(f)$, how to determine the roots of the

sextic $f(x)$, and how to explicitly determine the action of $\mathrm{Gal}(f)$ on the roots.

We first discuss criteria for determining the Galois group $\mathrm{Gal}(f)$. We recall that by Proposition 3, we know the value of the rational root $\theta_1 = \hat{\theta}_{48} = r_1 r_2 + r_3 r_4 + r_5 r_6$ of $f_{15}(x)$. We now introduce the variables

$$\begin{aligned} d_{12} &= x_1 + x_2, & d_{34} &= x_3 + x_4, & d_{56} &= x_5 + x_6, \\ e_{12} &= x_1 x_2, & e_{34} &= x_3 x_4, & e_{56} &= x_5 x_6, \end{aligned} \tag{8}$$

and

$$\chi_1 = (d_{12} - d_{34})(d_{34} - d_{56})(d_{56} - d_{12})$$

$$\chi_2 = (e_{12} - e_{34})(e_{34} - e_{56})(e_{56} - e_{12}).$$

Now χ_1^2, χ_2^2 are G_{48}-polynomials and by Proposition 3, the values of $\hat{\chi}_1^2$, $\hat{\chi}_2^2$ can be determined as roots of the Galois resolvents $f_{\chi_i^2}$. We now state some elementary properties of $\hat{\chi}_1$, $\hat{\chi}_2$.

LEMMA 4. *Let $f(x) \in \mathbf{Q}[x]$ be an irreducible sextic with* $\mathrm{Gal}(f) = G_{48}$, Γ_{24}, G_{24}, H_{24}, *or* Γ_{12}. *Then*

(a) $\hat{\chi}_1 = 0 \Leftrightarrow \hat{d}_{12} = \hat{d}_{34} = \hat{d}_{56} = a_1/3$.

(b) $\hat{\chi}_2 = 0 \Leftrightarrow \hat{e}_{12} = \hat{e}_{34} = \hat{e}_{56} = \theta_1/3$.

(c) *At least one of the $\hat{\chi}_i$ is non-zero. If both $\hat{\chi}_1$, $\hat{\chi}_2$ are non-zero, then $\hat{\chi}_1^2$ is a square in $\mathbf{Q} \Leftrightarrow \hat{\chi}_2^2$ is a square in $\mathbf{Q}$.*

Proof. We first prove (a). Suppose $\hat{\chi}_1 = 0$. Then one of the three factors of $\hat{\chi}_1$ must vanish. Assume that $\hat{d}_{12} = \hat{d}_{34}$. Then applying the automorphism $\sigma = (135)(246) \in \Gamma_{12} \subset G$, we obtain $\hat{d}_{34} = \hat{d}_{56}$, and $a_1 = 3\hat{d}_{12}$. Thus $(\Rightarrow)$ is proved. The converse $(\Leftarrow)$ is clear. Statement (b) is proved in the same way. We now prove (c). Assume that $\hat{\chi}_1 = \hat{\chi}_2 = 0$. Then by (a), (b), the symmetric functions in r_1 and r_2 are rational numbers. But then

$$x^2 - \hat{d}_{12} x + \hat{e}_{12} = (x - r_1)(x - r_2)$$

is a rational quadratic factor of $f(x)$, contradicting the irreducibility of $f(x)$. Hence, at least one of the $\hat{\chi}_i$ is non-zero. Now suppose that both are non-zero. Since χ_1/χ_2 is fixed by G_{48}, $\hat{\chi}_1/\hat{\chi}_2 \in \mathbf{Q}^*$ and (c) is proved. ∎

By Lemma 4, we can define a non-zero number $\chi \in \mathbf{Q}$ by $\chi = \hat{\chi}_1^2$ if $\hat{\chi}_1 \neq 0$ and $\chi = \hat{\chi}_2^2$ otherwise. We recall that Δ is the discriminant of $f(x)$. We now prove Theorem 3 of the Introduction and determine $\mathrm{Gal}(f)$.

Proof of Theorem 3. We first prove (b). Let σ denote the element $(12)(34)(56) \in G_{48}$. Then $\sigma(\Delta) = -\Delta$ and the stabilizer of Δ in G_{48} is Γ_{24}.

Since Δ is non-zero, we have that Δ is a rational square if and only if $\mathrm{Gal}(f) \subset \Gamma_{24}$. Since $\chi \neq 0$, $(13)(24)\chi = -\chi$ and $(13)(24)\chi\Delta = -\chi\Delta$, and G_{24} and H_{24} are the respective stabilizers of χ and $\chi\Delta$ in G_{48}, we obtain the corresponding statements (c), (d) for G_{24} and H_{24}. Cases (a) and (e) then follow from cases (b), (c), and (d) and the fact that $\Gamma_{12} = \Gamma_{24} \cap G_{24} \cap H_{24}$. ▮

We now show that there are general formulas for finding the roots of $f(x)$. Implicit in our approach will be the assumption that we can simplify algebraic numbers to determine whether they are rational numbers. We refer to [10] for a discussion of this problem. First, we introduce some useful symmetric functions of the d_{ij} and e_{ij}. Some of these symmetric functions can be easily expressed in terms of a_i, θ_1. We have

$$a_1 = \hat{d}_{12} + \hat{d}_{34} + \hat{d}_{56},$$

$$a_2 - \theta_1 = \hat{d}_{12}\hat{d}_{34} + \hat{d}_{34}\hat{d}_{56} + \hat{d}_{12}\hat{d}_{56},$$

and

$$\theta_1 = \hat{e}_{12} + \hat{e}_{34} + \hat{e}_{56},$$
$$a_6 = \hat{e}_{12}\hat{e}_{34}\hat{e}_{56}.$$

The other two symmetric functions, which are specializations of the two G_{48}-polynomials

$$\begin{aligned} D &= d_{12}d_{34}d_{56}, \\ E &= e_{12}e_{34} + e_{34}e_{56} + e_{12}e_{56}, \end{aligned} \tag{9}$$

are not so easily expressed. By Proposition 3, $\hat{D}$, $\hat{E}$ can be determined as rational roots of their Galois resolvents $\hat{F}_D(x)$, $\hat{F}_E(x) \in \mathbf{Q}[x]$. Let

$$g_2(x) = x^3 - a_1x^2 + (a_2 - \theta_1)x - \hat{D} \in \mathbf{Q}[x], \tag{10}$$

and

$$g_3(x) = x^3 - \theta_1x^2 + \hat{E}x - a_6 \in \mathbf{Q}[x]. \tag{11}$$

Let ω be a primitive cubic root of unity. Formulas $y_g(\omega^i)$ for finding the roots of a cubic polynomial g are given in Fig. 6 of Section 7. Define $l_i = y_{g_2}(\omega^i)$, $m_i = y_{g_3}(\omega^i)$, for $i = 1, 2, 3$. The l_i (resp. m_i) are the roots of $g_2(x)$ (resp. $g_3(x)$). We then have

$$\{l_1, l_2, l_3\} = \left\{\hat{d}_{12}, \hat{d}_{34}, \hat{d}_{56}\right\}$$

and

$$\{m_1, m_2, m_3\} = \{\hat{e}_{12}, \hat{e}_{34}, \hat{e}_{56}\}.$$

We note that we do not yet know how to identify the l_i (resp. m_i) with the $\hat{d}_{ij}$ (resp. $\hat{e}_{ij}$). Finally, let us define for $k = 1, 2$ the two G_{48}-polynomials:

$$h_{1k} = d_{12}^k e_{12} + d_{34}^k e_{34} + d_{56}^k e_{56}. \tag{12}$$

Since h_{11}, h_{12} are G_{48}-polynomials, $\hat{h}_{11}, h_{12} \in \mathbf{Q}$ when $\mathrm{Gal}(f) \subset G_{48}$.

PROPOSITION 7. *Let $f(x) \in \mathbf{Q}[x]$ be an irreducible sextic polynomial with $\mathrm{Gal}(f) \subset G_{48}$. Assume that the values of θ_1, l_i, m_i, $\hat{h}_{11}$, and $\hat{h}_{12}$ are known. Then there is an effective algorithm for determining $\sigma \in S_3$ such that for each i, l_i and $m_{\sigma(i)}$ correspond to the same pair of roots. In other words, we can find σ such that*

$$\{(l_i, m_{\sigma(i)})\}_{i=1,2,3} = \{(\hat{d}_{12}, \hat{e}_{12}), (\hat{d}_{34}, \hat{e}_{34}), (\hat{d}_{56}, \hat{e}_{56})\}. \tag{13}$$

Before reading through the proof of Proposition 7, the reader is urged to see its application in Theorems 7, 6. We now introduce some additional notation and prove a lemma needed to prove Proposition 7. Let $k = 1$ or 2. Define

$$\begin{aligned} & & j_{1k} &= d_{12}^k e_{12} + d_{34}^k e_{56} + d_{56}^k e_{34} \\ h_{2k} &= d_{12}^k e_{34} + d_{34}^k e_{56} + d_{56}^k e_{12}, & j_{2k} &= d_{12}^k e_{56} + d_{34}^k e_{34} + d_{56}^k e_{12} \\ h_{3k} &= d_{12}^k e_{56} + d_{34}^k e_{12} + d_{56}^k e_{34}, & j_{3k} &= d_{12}^k e_{34} + d_{34}^k e_{12} + d_{56}^k e_{56}. \end{aligned} \tag{14}$$

Fix $k = 1, 2$. Then for all groups G in Fig. 1 with $G \subset G_{48}$, the set $\{h_{2k}, h_{3k}\}$ is G-stable and the set $\{j_{1k}, j_{2k}, j_{3k}\}$ is a G-orbit in $\mathbf{Q}[x_1, \dots, x_6]$
Since $\mathrm{Gal}(f)$ satisfies this condition, we have

$$\hat{h}_{2k} \in \mathbf{Q} \Leftrightarrow \hat{h}_{3k} \in \mathbf{Q}, \tag{15}$$

and

$$\hat{j}_{ik} \in \mathbf{Q} \text{ for some } i \Leftrightarrow \hat{j}_{ik} \in \mathbf{Q} \text{ for all } i. \tag{16}$$

When the last case occurs, $\hat{j}_{1k} = \hat{j}_{2k} = \hat{j}_{3k}$.

LEMMA 5. *Let $f(x) \in \mathbf{Q}[x]$ be an irreducible sextic polynomial with $\mathrm{Gal}(f) \subset G_{48}$. Let l_i, m_i, h_{ik}, j_{ik} be defined as above. Fix $k = 1, 2$. If the l_i*

are distinct and the m_i *are distinct, then*

$$\hat{h}_{1k} \notin \left\{\hat{j}_{1k}, \hat{j}_{2k}, \hat{j}_{3k}\right\}.$$

Proof. We first assume that $k = 1$. If $\hat{h}_{11} = \hat{j}_{11}$, then $(\hat{d}_{34} - \hat{d}_{56})(\hat{e}_{34} - \hat{e}_{56}) = 0$ and $\hat{d}_{34} = \hat{d}_{56}$ or $\hat{e}_{34} = \hat{e}_{56}$. But this contradicts either the distinctness of the l_i or that of the m_i. Hence $\hat{h}_{11} \neq \hat{j}_{11}$. The other cases are proved similarly. Now suppose that $k = 2$ and that $\hat{h}_{12}$ equals one of the $\hat{j}_{i2}$. Then $\hat{j}_{i2} \in \mathbf{Q}$ for $i = 1, 2, 3$. Hence $\hat{j}_{12} = \hat{j}_{22} = \hat{j}_{32}$. Now the same argument as when $k = 1$ shows that $\hat{h}_{12} = \hat{j}_{12}$ implies that $\hat{d}_{34} = -\hat{d}_{56}$. Similarly $\hat{h}_{12} = \hat{j}_{22}$ implies that $\hat{d}_{12} = -\hat{d}_{56}$. Hence $\hat{d}_{12} = \hat{d}_{34}$ and the l_i are not distinct, contradicting the hypothesis. Hence the lemma is proved. ∎

Finally, for $k = 1, 2$, $\sigma \in S_3$, define

$$p_{k\sigma} = l_1^k m_{\sigma(1)} + l_2^k m_{\sigma(2)} + l_3^k m_{\sigma(3)}. \tag{17}$$

The $p_{k\sigma}$ have the property that $\{p_{k\sigma} \mid \sigma \in S_3\} = \{\hat{h}_{ik}, \hat{j}_{ik} \mid i = 1, 2, 3\}$. The following lemma is essential.

LEMMA 6. *Let* $f(x) \in \mathbf{Q}[x]$ *be an irreducible sextic polynomial with* $\mathrm{Gal}(f) \subset G_{48}$. *Let* l_i, m_i, $p_{k\sigma}$ *be defined as above. Assume that the* l_i *are distinct and the* m_i *are distinct. Then there exists a unique* $\sigma \in S_3$ *such that* $p_{1\sigma} = \hat{h}_{11}$ *and* $p_{2\sigma} = \hat{h}_{12}$.

Proof. By definition of l_i, m_i, the equations $p_{j\sigma} = \hat{h}_{1j}$, for $j = 1, 2$, have at least one solution σ. We now establish uniqueness. Assume that we have σ_1, $\sigma_2 \in S_3$ with $p_{k\sigma_1} = p_{k\sigma_2} = \hat{h}_{1k}$ for $k = 1, 2$. Then the three equations

$$\begin{aligned} x + y + z &= 0 \\ l_1 x + l_2 y + l_3 z &= 0 \\ l_1^2 x + l_2^2 y + l_3^2 z &= 0 \end{aligned} \tag{18}$$

have the non-zero solution

$$(x, y, z) = (m_{\sigma_1(1)} - m_{\sigma_2(1)}, m_{\sigma_1(2)} - m_{\sigma_2(2)}, m_{\sigma_1(3)} - m_{\sigma_2(3)}).$$

But since the determinant

$$\Delta' = \begin{vmatrix} 1 & 1 & 1 \\ l_1 & l_2 & l_3 \\ l_1^2 & l_2^2 & l_3^2 \end{vmatrix} = -\prod_{i<j} (l_i - l_j)$$

Formulas for the Roots of f when $\mathrm{Gal}(f) \subset G_{48}$, $\mathrm{Gal}(f) \not\subset G_{72}$.

1. Let $f(x) = x^6 - a_1x^5 + a_2x^4 - a_3x^3 + a_4x^2 - a_5x + a_6$.
2. Use Theorem 2 to determine $\mathrm{Gal}(f) \subset G_{48}$ and $\mathrm{Gal}(f) \not\subset G_{72}$. Let θ_1 be the unique rational root of $f_{15}(x)$ with multiplicity 1, 5, 7, or 9.
3. For $z = D$, E, h_{11}, h_{12} defined as in (8), (9), (12), calculate the G_{48}-resolvent $F_z(x)$. For each z, let $\hat{z} \in \mathbf{Z}$ be the unique rational root of $\hat{F}_z(x)$ with multiplicity 1, 5, 7, or 9.
4. Let ω be a primitive cubic root of unity and let $y_g(\omega^i)$ be the formulas in Figure 6.
5. Let $l_i = y_{g_2}(\omega^i)$ be the three roots of the cubic polynomial $g_2 = x^3 - a_1x^2 + (a_2 - \theta_1)x - \hat{D}$ and let $m_i = y_{g_3}(\omega^i)$ be the three roots of the cubic polynomial $g_3 = x^3 - \theta_1x^2 + \hat{E}x - a_6$.
6. For $k = 1, 2$, $\sigma \in S_3$, define $p_{k\sigma} = \sum_{i=1}^{3} l_i^k m_{\sigma(i)}$.
7. a. If $l_i = l_j$ or $m_i = m_j$ for some $i \neq j$, let $\sigma = 1$.

   b. Otherwise, let σ be the unique element of S_3 with $p_{1\sigma} = \hat{h}_{11}$, $p_{2\sigma} = \hat{h}_{12}$.
8. Define $z(f, a, b, \varepsilon) = \frac{1}{2}\left(y_{g_2}(a) + \varepsilon\sqrt{y_{g_2}(a)^2 - 4y_{g_3}(b)}\right)$. Let $z_{2i-1} = z(f, \omega^i, \omega^{\sigma(i)}, 1)$ for $i = 1, 2, 3$ and $z_{2i} = z(f, \omega^i, \omega^{\sigma(i)}, -1)$ for $i = 1, 2, 3$.
9. The z_i are formulas for the roots of $f(x)$ in the variables a_i, θ_1, $\hat{D}$, $\hat{E}$. The formulas use only the basic arithmetic operations and radicals. The Galois action of $\tau \in \mathrm{Gal}(f)$ on the z_i is given by $\tau(z_i) = z_{\tau(i)}$.

FIG. 4. Formulas for finding the roots of an irreducible, sextic $f(x) \in \mathbf{Q}[x]$ when $\mathrm{Gal}(f) \subset G_{48}$, $\mathrm{Gal}(f) \not\subset G_{72}$.

is non-zero as the l_i are distinct, the only solution to (18) is the trivial solution. Since the m_i are distinct, we have $\sigma_1 = \sigma_2$ and the lemma is proved. ∎

We can now prove Proposition 7. The steps in the algorithm are listed in Steps 6 and 7 of Fig. 4.

Proof of Proposition 7. We first note that (13) is satisfied for at least one $\sigma \in S_3$, and to prove the proposition, we need only show how to determine σ. We first consider the case when two of the l_i coincide. Then the action of $(135)(246) \in \mathrm{Gal}(f)$ shows that $l_1 = l_2 = l_3$. Similarly, if two of the m_i coincide, then they are all equal. In either case, the proposition holds trivially by letting $\sigma = (1)$. We now consider the case when the l_i are distinct and the m_i are distinct. Now (13) has at least one solution $\sigma \in S_3$ and any solution is a solution to the equations $p_{1\sigma} = \hat{h}_{11}$ and $p_{2\sigma} = \hat{h}_{12}$.

By Lemma 6, these equations have a unique solution. Hence by comparing the values of $p_{j\sigma}$ to those of the known constants $\hat{h}_{11}$, $\hat{h}_{12}$, we can determine $\sigma \in S_3$ satisfying (13) and the proposition is proved. ∎

Proposition 7 is the key to solving the sextic when $\mathrm{Gal}(f) \subset G_{48}$, $\mathrm{Gal}(f) \not\subset G_{72}$.

THEOREM 6. *Let G be one of the transitive, solvable subgroups G_{48}, Γ_{24}, G_{24}, H_{24}, Γ_{12} of S_6. Let $\overline{G}$ be its conjugacy class in Σ_6.*

(a) *The general equation of type $(6, \overline{G})$ is explicitly solvable by radicals.*

(b) *The formulas $z_i(t_j)$ in* (a) *can be numbered so that for each $f \in \Gamma_{\overline{G}}$, the Galois action of $\tau \in \mathrm{Gal}(f)$ on the roots $z_i = z_i(\hat{t}_j(f))$ is given by $\tau(z_i) = z_{\tau(i)}$.*

Proof. (a) Given an irreducible sextic polynomial $f \in \Gamma_{\overline{G}}$ with $\mathrm{Gal}(f) = G$, define the polynomials g_2, g_3 as in (10), (11). Let $y_g(a)$ be defined as in Fig. 6 of Section 7, ω a primitive cubic root of unity and define $z(f, a, b, \varepsilon) = \frac{1}{2}\left(y_{g_2}(a) + \varepsilon\sqrt{y_{g_2}(a)^2 - 4y_{g_3}(b)}\,\right)$. Let l_i, m_i be the roots of g_2, g_3 defined following (11). By Proposition 3, we can determine the values of $\theta_1 = \hat{\theta}_{48}$, $\hat{h}_{11}$, $\hat{h}_{12}$. By Proposition 7, one can calculate $\sigma \in S_3$ such that (13) holds. Define

$$z_{2i-1} = z\left(f, \omega^i, \omega^{\sigma(i)}, 1\right), \qquad z_{2i} = z\left(f, \omega^i, \omega^{\sigma(i)}, -1\right), \text{ for } i = 1, 2, 3. \tag{19}$$

Then

$$\{\{r_1, r_2\}, \{r_3, r_4\}, \{r_5, r_6\}\} = \{\{z_1, z_2\}, \{z_3, z_4\}, \{z_5, z_6\}\}. \tag{20}$$

Hence $z(f, a, b, \varepsilon)$ provides formulas for the roots $\{z_i\}$ of f in terms of the variables a, b, ε, $\hat{D}$, $\hat{E}$, and θ_1. Since there is a finite algorithm for calculating their values given f and all values in $K = \mathbf{Q}[\omega]$, (a) is proved.

The proof of (b) is the same as that of Proposition 6(b). To prove (b), it suffices to show that for any z_i arising from the formulas in (a), there is an automorphism $\alpha : \mathrm{Gal}(f) \to \mathrm{Gal}(f)$ satisfying $\alpha(\sigma)(z_i) = z_{\sigma(i)}$, for $\sigma \in \mathrm{Gal}(f)$. Then, by twisting the Galois action by α, (b) holds. Now the roots r_i were initially chosen so that $\mathrm{Gal}(f)$ was one of the five subgroups G_{48}, G_{24}, Γ_{24}, H_{24}, Γ_{12} and have the property that $\sigma(r_i) = r_{\sigma(i)}$ for $\sigma \in \mathrm{Gal}(f)$. Now the proof of (a) shows (20) holds. Hence, there exists $\tau \in G_{48}$ such that $z_i = r_{\tau(i)}$ and consequently, $\sigma(z_i) = z_{\tau^{-1}\sigma\tau(i)}$ for $\sigma \in \mathrm{Gal}(f)$. Since τ normalizes each of the groups G_{48}, G_{24}, Γ_{24}, H_{24}, and Γ_{12}, we have $\tau\sigma\tau^{-1} \in \mathrm{Gal}(f)$ and $(\tau\sigma\tau^{-1})(z_i) = z_{\sigma(i)}$. Hence, letting $\alpha(\sigma) = \tau\sigma\tau^{-1}$ gives the desired map. ∎

The following lemma, which will be used in Section 6, follows trivially from the proof of Theorem 6.

LEMMA 7. *Let $G \subset G_{48}$ be one of the transitive groups in Fig.* 1 *and let $\overline{G}$ be its conjugacy class in Σ_6. Suppose that $f \in \Gamma_{\overline{G}}$ is an irreducible sextic with* $\mathrm{Gal}(f) = G$, *let r_i be the corresponding numbering of the roots, and suppose that the values of θ_1, $\hat{D}$, $\hat{E}$, l_i, m_i, $\hat{h}_{11}$, $\hat{h}_{12}$ are known. Let $\sigma \in S_3$ be the element determined by Proposition* 7 *and let z_i be defined as in* (19). *Then*

$$\{\{r_1, r_2\}, \{r_3, r_4\}, \{r_5, r_6\}\} = \{\{z_1, z_2\}, \{z_3, z_4\}, \{z_5, z_6\}\}.$$

6. SOLVING THE SEXTIC: $\mathrm{Gal}(f) \subset G_{12}$

We now consider the case when $\mathrm{Gal}(f) \subset G_{12}$. We assume that Theorem 2 has already been used to determine that $\mathrm{Gal}(f) \subset G_{12}$. As $\mathrm{Gal}(f) \subset G_{72}$, Proposition 6 shows that there are formulas for finding the roots z_i of $f(x)$. We now establish criteria for determining $\mathrm{Gal}(f)$, and its explicit action on the roots when $\mathrm{Gal}(f) = G_{12}$, C_6, or H_6.

We first determine whether $\mathrm{Gal}(f) = G_{12}$, C_6, or H_6. In practice, inspection may show that the roots r_i generate a degree 12 extension of $\mathbf{Q}$ and thus $\mathrm{Gal}(f) = G_{12}$. We now present a general method.

We recall the G_{72}-polynomials α_i^2, γ_i, M, N, M_1, N_1 defined in (4), (2), and (5). By Proposition 3, their specializations with respect to f can be determined as $G \subset G_{72}$. Define the polynomials $g(x)$, $g_1(x)$ as in (3), (6). We recall that the roots of $g(x)$ are $\hat{\delta}^2$, $\hat{\mu}^2$ and the roots of $g_1(x)$ are $\hat{\delta}_1^2$, $\hat{\mu}_1^2$, where δ^2, δ_1^2, μ^2, μ_1^2 are G_{36}-polynomials defined in (2), (5). Since $\mathrm{Gal}(f) \subset G_{36}$, the specializations $\hat{\delta}^2$, $\hat{\mu}^2$, $\hat{\delta}_1^2$, $\hat{\mu}_1^2$ are rational. We can now present a criterion for calculating the Galois group.

THEOREM 7. *Let $f(x) \in \mathbf{Q}[x]$ be an irreducible sextic polynomial with* $\mathrm{Gal}(f) \subset G_{12}$. *Then*

(a) $\mathrm{Gal}(f) = G_{12} \Leftrightarrow \hat{\delta}^2$, $\hat{\delta}_1^2$, $\hat{\mu}^2$, $\hat{\mu}_1^2$ *are each zero or a non-square in* $\mathbf{Q}$.

(b) $\mathrm{Gal}(f) = C_6 \Leftrightarrow$ *either $\hat{\delta}^2$ or $\hat{\delta}_1^2$ is a non-zero square in* $\mathbf{Q}$.

(c) $\mathrm{Gal}(f) = H_6 \Leftrightarrow$ *either $\hat{\mu}^2$ or $\hat{\mu}_1^2$ is a non-zero square in* $\mathbf{Q}$.

Proof. By Lemma 3, either $\hat{\delta}^2$ or $\hat{\delta}_1^2$ is non-zero and either $\hat{\mu}^2$ or $\hat{\mu}_1^2$ is non-zero. We now note that δ, δ_1 are invariant under C_6 and the action of $\tau = (13)(24) \in G_{12}$ is $\tau(\delta) = -\delta$, $\tau(\delta_1) = -\delta_1$. Also, μ, μ_1 are invariant under H_6 and the action of $\sigma = (12)(34)(56) \in G_{12}$ is $\sigma(\mu) = -\mu$, $\sigma(\mu_1) = -\mu_1$. The statements (a)–(c) are easy consequences. ∎

The proof of Theorem 7 also establishes

COROLLARY 2. *Let $f(x) \in \mathbf{Q}[x]$ be an irreducible sextic polynomial with* $\mathrm{Gal}(f) \subset G_{12}$. *Assume that the numbers $\hat{\delta}^2$, $\hat{\mu}^2$ are non-zero. Then*

(a) $\mathrm{Gal}(f) = G_{12} \Leftrightarrow \hat{\delta}^2$, $\hat{\mu}^2$ *are non-squares in* $\mathbf{Q}$.

(b) $\mathrm{Gal}(f) = C_6 \Leftrightarrow \hat{\delta}^2$ *is a square in* $\mathbf{Q}$.

(c) $\mathrm{Gal}(f) = H_6 \Leftrightarrow \hat{\mu}^2$ *is a square in* $\mathbf{Q}$.

The astute reader will note that the criterion in both Theorem 7 and Corollary 2 for determining $\mathrm{Gal}(f)$ depends upon being able to determine the values of the specializations $\hat{\delta}^2$, $\hat{\mu}^2$, $\hat{\delta}_1^2$, $\hat{\mu}_1^2$. Now while we can determine $\{\hat{\delta}^2, \hat{\mu}^2\}$ as the set of rational roots of $g(x)$, we cannot distinguish $\hat{\delta}^2$ from $\hat{\mu}^2$. We have the same problem with $\hat{\delta}_1^2$ and $\hat{\mu}_1^2$. We now show how to identify the roots of $g(x)$ and $g_1(x)$ and make the criteria in the above theorem and corollary effective.

Let H denote the group $H = \{e, (24), (26), (46), (246), (264)\} \cong S_3$. For each $\sigma \in H$ define

$$m_\sigma = x_1 x_{\sigma(2)} + x_3 x_{\sigma(4)} + x_5 x_{\sigma(6)} \in \mathbf{Q}[x_1, \dots, x_6].$$

Since m_e is a G_{48}-polynomial, $\hat{m}_e \in \mathbf{Q}$. For the groups $G = G_{12}, C_6, H_6$, the set $\{m_{(246)}, m_{(264)}\}$ is G-stable and $\{m_{(24)}, m_{(26)}, m_{(46)}\}$ is a G-orbit. Thus

$$\hat{m}_{(246)} \in \mathbf{Q} \Leftrightarrow \hat{m}_{(264)} \in \mathbf{Q}, \tag{21}$$

and

$$\hat{m}_\sigma \in \mathbf{Q} \text{ for some } \sigma \in \{(24), (26), (46)\} \Leftrightarrow \hat{m}_{(24)} = \hat{m}_{(26)} = \hat{m}_{(46)} \in \mathbf{Q}. \tag{22}$$

We have

LEMMA 8. *Let $f(x) \in \mathbf{Q}[x]$ be an irreducible sextic polynomial and let m_σ be defined as above. Then $\{\hat{m}_e, \hat{m}_{(246)}, \hat{m}_{(264)}\} \cap \{\hat{m}_{(24)}, \hat{m}_{(26)}, \hat{m}_{(46)}\} = \emptyset$.*

Proof. Assume $\hat{m}_e = \hat{m}_{(46)}$. Then $(r_3 - r_5)(r_4 - r_6) = 0$ and thus $r_3 = r_5$ or $r_4 = r_6$, which cannot occur. The other cases are similar. ∎

We now recall that given an irreducible sextic $f(x) \in \mathbf{Q}[x]$, the r_i are roots of $f(x)$ that were fixed in order to calculate the specialization with respect to $f(x)$. Since $\mathrm{Gal}(f) \subset G_{72}$, Proposition 6 provides formulas z_i for the roots of $f(x)$. The formulas require that the r_i were numbered so that $\mathrm{Gal}(f) \subset G_{72}$, but are otherwise independent of the specific numbering of r_i. Equation (7) in the proof of Proposition 6 shows that

$$\{\{z_1, z_3, z_5\}, \{z_2, z_4, z_6\}\} = \{\{r_1, r_3, r_5\}, \{r_2, r_4, r_6\}\}.$$

Now the elements (12)(34)(56), (135)(246), (13)(24) $\in S_6$ normalize G_{12} and its subgroups. Hence we can use them to twist the Galois action and renumber the r_i without changing $\mathrm{Gal}(f)$. We can thus assume that

$$(z_1, z_3, z_5) = (r_1, r_3, r_5) \qquad \text{and} \qquad \{z_2, z_4, z_6\} = \{r_2, r_4, r_6\}. \quad (23)$$

With H as above, for each $\sigma \in H$, we define

$$n_\sigma = z_1 z_{\sigma(2)} + z_3 z_{\sigma(4)} + z_5 z_{\sigma(6)}.$$

We have $\{n_\sigma\}_{\sigma \in H} = \{\hat{m}_\sigma\}_{\sigma \in H}$. It will not be true in general that $n_\sigma = \hat{m}_\sigma$.

LEMMA 9. *Let $f(x) \in \mathbf{Q}[x]$ be an irreducible sextic polynomial with $\mathrm{Gal}(f) \subset G_{12}$ and let n_σ, m_σ be defined as above. Then at least one of the following statements holds.*

(a) *There is a unique $\sigma \in H$ such that $n_\sigma = \hat{m}_e$. When this occurs, there is an algorithm for calculating σ.*

(b) *One can calculate $\sigma \in H$ such that $n_\sigma, n_{\sigma(246)}, n_{\sigma(264)} \in \mathbf{Q}$ and*

$$\{n_\sigma, n_{\sigma(246)}, n_{\sigma(264)}\} = \left\{\hat{m}_e, \hat{m}_{(246)}, \hat{m}_{(264)}\right\}.$$

Proof. Equations (21), (22) show that $n_\sigma \in \mathbf{Q}$ for exactly 1, 3, 4 or 6 values of $\sigma \in H$. If n_σ is rational for exactly one $\sigma \in H$, then since $\hat{m}_e \in \mathbf{Q}$, we hve $n_\sigma = \hat{m}_e$. By construction, σ is known. Hence statement (a) holds. If n_σ is rational for exactly 3 values $\sigma = \sigma_1, \sigma_2, \sigma_3$, then

$$\{n_{\sigma_1}, n_{\sigma_2}, n_{\sigma_3}\} = \left\{\hat{m}_e, \hat{m}_{(246)}, \hat{m}_{(264)}\right\}.$$

By (21) and the definition of n_σ, we have $\{\sigma_1, \sigma_2, \sigma_3\} = \{\sigma_1, \sigma_1(246), \sigma_1(264)\}$. Hence, letting $\sigma = \sigma_1$, statement (b) is satisfied. Similarly, if n_σ is rational for exactly 4 values, then (21), (22) show that $\hat{m}_{(24)} = \hat{m}_{(26)} = \hat{m}_{(46)} \in \mathbf{Q}$ and that $\hat{m}_{(246)}, \hat{m}_{(264)}$ are irrational. By Lemma 8, $\hat{m}_e$ is then the unique rational value appearing with multiplicity one among the n_σ. Hence statement (a) holds and σ is easily determined. Finally, if $n_\sigma \in \mathbf{Q}$ for all $\sigma \in H$, then $\hat{m}_{(24)} = \hat{m}_{(26)} = \hat{m}_{(46)}$. By Lemma 8, the value $\hat{m}_{(24)}$ appears with multiplicity three in $\{n_\sigma\}$.

If there exists $\sigma_1 \in S_3$ with n_{σ_1} appearing with multiplicity one or two in the set $\{n_\sigma\}$, then letting $\sigma = \sigma_1$, (b) is satisfied. If such a σ_1 does not exist, we must have $\hat{m}_e = \hat{m}_{(246)} = \hat{m}_{(264)}$. As $\hat{m}_e + \hat{m}_{(246)} + \hat{m}_{(264)} = \hat{m}_{(24)} + \hat{m}_{(26)} + \hat{m}_{(46)}$ by the definition of m_σ, we have

$$3\hat{m}_e = \hat{m}_e + \hat{m}_{(246)} + \hat{m}_{(264)} = \hat{m}_{(24)} + \hat{m}_{(26)} + \hat{m}_{(46)} = 3\hat{m}_{(24)},$$

which contradicts Lemma 8. Hence this cannot occur and the lemma is proved. ∎

We can now show

PROPOSITION 8. *Let $f(x) \in \mathbf{Q}[x]$ be an irreducible sextic polynomial such that* $\mathrm{Gal}(f) \subset G_{12}$. *Let δ, δ_1, μ, μ_1 be defined as in* (2), (5). *Then $\hat{\delta}^2$, $\hat{\delta}_1^2$, $\hat{\mu}^2$, and $\hat{\mu}_1^2$ can be computed.*

Proof. We first assume that statement (a) of Lemma 9 holds and that $\sigma \in H$ is the unique element in H such that $n_\sigma = \hat{m}_e$. Thus,

$$\{(r_i, r_{i+1})\}_{i=1,3,5} = \{(z_i, z_{\sigma(i+1)})\}_{i=1,3,5}.$$

The definition of δ shows that

$$\hat{\delta}^2 = \Big[(z_1 - z_3)(z_3 - z_5)(z_5 - z_1) + (z_{\sigma(2)} - z_{\sigma(4)})(z_{\sigma(4)} - z_{\sigma(6)})(z_{\sigma(6)} - z_{\sigma(2)})\Big]^2 \tag{24}$$

and

$$\hat{\mu}^2 = \Big[(z_1 - z_3)(z_3 - z_5)(z_5 - z_1) - (z_{\sigma(2)} - z_{\sigma(4)})(z_{\sigma(4)} - z_{\sigma(6)})(z_{\sigma(6)} - z_{\sigma(2)})\Big]^2. \tag{25}$$

Similarly, one can use σ to calculate $\hat{\delta}_1^2$, $\hat{\mu}_1^2$. Now assume that statement (b) of Lemma 9 holds and let $\sigma \in H$ be the element given by the lemma. Then

$$\{(r_i, r_{i+1})\}_{i=1,3,5} = \{(z_i, z_{\sigma\tau(i+1)})\}_{i=1,3,5}, \tag{26}$$

for some $\tau \in \{e, (246), (264)\}$. Regardless of the choice of τ, one can check that formulas (24), (25) hold and $\hat{\delta}^2$, $\hat{\mu}^2$ can be calculated. Similarly, we can calculate $\hat{\delta}_1^2$, $\hat{\mu}_1^2$. The proposition is proved. ∎

Finally, we show how to explicitly exhibit the Galois action of $\mathrm{Gal}(f)$ on the roots z_i of f (see Fig. 5). Unlike in Sections 4, 5, the algorithm will depend upon $\mathrm{Gal}(f)$.

PROPOSITION 9. *Let $G = G_{12}, C_6$, or H_6 and suppose $f(x) \in \mathbf{Q}[x]$ is an irreducible sextic polynomial with $\mathrm{Gal}(f) = G$. Let z_i be the roots of $f(x)$ found by the formula in Proposition* 6. *There is a bounded algorithm for renumbering the z_i so that $\tau \in \mathrm{Gal}(f) \subset S_6$ acts via $\tau(z_i) = z_{\tau(i)}$.*

Proof. From (23), we can assume that

$$(z_1, z_3, z_5) = (r_1, r_3, r_5) \qquad \text{and} \qquad \{z_2, z_4, z_6\} = \{r_2, r_4, r_6\}.$$

1. Find the roots z_i of the polynomial $f(x) = x^6 - a_1x^5 + a_2x^4 - a_3x^3 + a_4x^2 - a_5x + a_6$ by the algorithm described in Figure 3.
2. For $\sigma \in H = \{e, (24), (26), (46), (246), (264)\}$, define $n_{\sigma,0} = z_1 z_{\sigma(2)} + z_3 z_{\sigma(4)} + z_5 z_{\sigma(6)}$. Define $n_{\sigma,i}$ for $i = 1, 4$ as in (28).
3. If there exists $\sigma \in H$ and $0 \leq i \leq 4$ such that $n_{\sigma,i}$ is the unique integer occurring with multiplicity one in the set $\{n_{\tau,i} | \tau \in H\}$, let $z_j'' = z_{\sigma(j)}$, for $j = 0, 1, \ldots, 4$.
4. a. Otherwise, choose $\sigma \in H$ such that $n_{\sigma,0} = n_{\sigma(246),0} = n_{\sigma(264),0} \in \mathbf{Q}$. Let $\theta_1 = n_{\sigma,0}$, $\hat{D} = n_{\sigma,1}$, $\hat{E} = n_{\sigma,3}$, $\hat{h}_{11} = n_{\sigma,3}$, $\hat{h}_{12} = n_{\sigma,4}$. Then let z_i' be the roots found by following Steps 4-9 in the algorithm in Figure 4.

   b. Use $\{\{z_1, z_3, z_5\}, \{z_2, z_4, z_6\}\} = \{\{r_1, r_3, r_5\}, \{r_2, r_4, r_6\}\}$ and
$$\{\{z_1', z_2'\}, \{z_3', z_4'\}, \{z_5', z_6'\}\} = \{\{r_1, r_2\}, \{r_3, r_4\}, \{r_5, r_6\}\}$$
   to define z_i'' such that
$$\{(z_1'', z_3'', z_5''), (z_2'', z_4'', z_6'')\} = \{(r_1, r_3, r_5), (r_2, r_4, r_6)\}$$
5. The z_i'' are the roots of $f(x)$ and the Galois action of $\tau \in \mathrm{Gal}(f)$ on the z_i'' is given by $\tau(z_i'') = z_{\tau(i)}''$.

FIG. 5. The algorithm for determining the explicit Galois action of $\mathrm{Gal}(f)$ on the roots of $f(x)$ when $\mathrm{Gal}(f) \subset G_{12}$.

If statement Lemma 9(a) holds, then $r_i = z_{\sigma(i)}$, for all i, for some known $\sigma \in H$. Hence we can renumber the z_i using σ so that $z_i = r_i$ for all i. Then $\tau \in \mathrm{Gal}(f)$ acts via $\tau(z_i) = z_{\tau(i)}$ and the proposition is proved. We now suppose that only statement Lemma 9(b) holds. We can then renumber the z_i so that

$$z_i = r_{\sigma(i)}, \tag{27}$$

for some $\sigma \in \{e, (246), (264)\}$. Now suppose that $\mathrm{Gal}(f) = H_6$. Since H_6 is normalized by (246), we can twist the Galois action by conjugating by a power of (246) without changing $\mathrm{Gal}(f)$. Hence we can assume that the r_i were chosen so that $z_i = r_i$ for all i and this choice establishes the proposition when $\mathrm{Gal}(f) = H_6$.

We now assume that $\mathrm{Gal}(f) = G_{12}$ or C_6 and that only statement (b) of Lemma 9 holds true. The argument we used when $\mathrm{Gal}(f) = H_6$ does not now work as (246) is not in the normalizer of G_{12} and C_6. Hence renumbering the r_i using (246) would change the Galois group through conjugation. When $\mathrm{Gal}(f) = G_{12}$ or C_6, we have the following lemma.

LEMMA 10. *Let $f(x) \in \mathbf{Q}[x]$ be an irreducible sextic polynomial with* $\mathrm{Gal}(f) = G_{12}, C_6$. *Let $\zeta_1 \in \mathbf{Q}[x_1, \dots, x_6]$ be a G_{48}-polynomial and let $\zeta_2 = (246)\zeta_1$, $\zeta_3 = (264)\zeta_1$ be its conjugates under the S_6-action. Then either*

(a) *$\hat{\zeta}_1$ is the unique integer appearing with multiplicity one in the set $\{\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3\}$, or*

(b) $\hat{\zeta}_1 = \hat{\zeta}_2 = \hat{\zeta}_3 \in \mathbf{Z}$.

Proof. The polynomials ζ_2, ζ_3 are conjugate under $G = G_{12}, C_6$. Hence $\hat{\zeta}_2$ and $\hat{\zeta}_3$ are conjugate under $\mathrm{Gal}(f)$. Since the ζ_i are algebraic integers, the lemma is an immediate consequence. ∎

We now apply the lemma to $\zeta_1 = m_e$. Then $\zeta_2 = m_{(246)}$, $\zeta_3 = m_{(264)}$. When Lemma 10(a) holds, $\{n_e, n_{(246)}, n_{(264)}\}$ contains a unique integer with multiplicity one, and we can determine $\sigma \in \{e, (246), (264)\}$, such that $\hat{m}_e = n_\sigma$. Since σ is uniquely determined, we have $r_i = z_{\sigma(i)}$ for all i. Hence we can renumber the z_i using σ so that $z_i = r_i$ for all i. When Lemma 10(b) holds, then $n_e = n_{(246)} = n_{(264)}$ and the value of $\theta_1 = \hat{\theta}_{48} = \hat{m}_e = \hat{\zeta}_i$ is known.

We can apply the same logic when ζ_1 equals each of the elements D, E, h_{11}, and h_{12} defined in Section 5. In each case, we can calculate the set $\{\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3\}$ using z_i and the respective functions

$$\begin{aligned}
n_{\sigma,1} &= (z_1 + z_{\sigma(2)})(z_3 + z_{\sigma(4)})(z_5 + z_{\sigma(6)}),\\
n_{\sigma,2} &= z_1 z_{\sigma(2)} z_3 z_{\sigma(4)} + z_1 z_{\sigma(2)} z_5 z_{\sigma(6)} + z_3 z_{\sigma(4)} z_5 z_{\sigma(6)},\\
n_{\sigma,3} &= z_1 z_{\sigma(2)}(z_1 + z_{\sigma(2)}) + z_3 z_{\sigma(4)}(z_3 + z_{\sigma(4)}) + z_5 z_{\sigma(6)}(z_5 + z_{\sigma(6)}),
\end{aligned} \tag{28}$$

and

$$n_{\sigma,4} = z_1 z_{\sigma(2)}(z_1 + z_{\sigma(2)})^2 + z_3 z_{\sigma(4)}(z_3 + z_{\sigma(4)})^2 + z_5 z_{\sigma(6)}(z_5 + z_{\sigma(6)})^2,$$

for $\sigma = e$, (246), (264). If for any of these choices of ζ_1, case (a) of Lemma 10 holds, then as in the previous paragraph, we can determine how to renumber the z_i so that $r_i = z_i$ for all i. Otherwise, if case (b) of Lemma 10 holds for all choices of ζ_1, we can then determine the values of $\hat{D}$, $\hat{E}$, $\hat{h}_{11}$, $\hat{h}_{12}$. Since θ_1 was already determined above, we can then determine the polynomials $g_2(x)$, $g_3(x)$ in (10), (11). Let $\{l_i\}$ be the set of roots of $g_2(x)$ and let $\{m_i\}$ be the set of roots of $g_3(x)$. Hence, the hypotheses of Lemma 7 are satisfied and we can determine the unique $\sigma \in \{e, (246), (264)\}$ satisfying

$$\{\{r_1, r_2\}, \{r_3, r_4\}, \{r_5, r_6\}\} = \{\{z_1, z_{\sigma(2)}\}, \{z_3, z_{\sigma(4)}\}, \{z_5, z_{\sigma(6)}\}\}.$$

Combined with (23), we can then renumber the z_i such that $r_i = z_i$ for all i. Hence in all cases, we can determine how to renumber the z_i so that $r_i = z_i$ for all i. Then the $\tau \in \mathrm{Gal}(f)$ acts on z_i by $\tau(z_i) = z_{\tau(i)}$ for $\sigma \in \mathrm{Gal}(f)$. The proposition is proved. ∎

7. EXAMPLES

We now give three examples of irreducible sextic polynomials $f \in \mathbf{Q}[x]$ where we calculate $\mathrm{Gal}(f)$, the roots z_i of f, and the Galois action on the z_i.

EXAMPLE 1. Let $f(x)$ be the cyclotomic polynomial

$$f(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1.$$

Let $\omega_7 = e^{2\pi i/7}$ be a primitive 7th root of unity. It is well known that $\mathrm{Gal}(f) = C_6 = \mathbf{Z}/6\mathbf{Z}$ and is generated by $\omega_7 \mapsto \omega_7^3$. We will recover these facts while finding the roots of $f(x)$. We first calculate the resolvent polynomials $f_{10}(x)$, $f_{15}(x)$. The formulas in the Appendix show that $f_{10}(x)$ is the polynomial

$$x^{10} - 6x^9 + 12x^8 - 64x^7 + 175x^6 - 224x^5 + 259x^4 - 170x^3 + 124x^2 - 80x + 16.$$

It has a unique rational root $x = 2$ and factors into irreducible polynomials as

$$f_{10}(x) = (x-2)(x^3 - 6x^2 + 5x - 1) \times (x^6 + 2x^5 + 11x^4 + x^3 + 16x^2 + 4x + 8).$$

From the Appendix, we also have

$$f_{15}(x) = x^{15} - 3x^{14} - 28x^{12} + 91x^{11} - 7x^{10} + 147x^9 - 664x^8 + 193x^7 + 98x^6 + 770x^5 - 77x^4 - 1645x^3 + 539x^2 + 611x - 258.$$

$f_{15}(x)$ also has $x = 2$ as its only rational root and has the irreducible factorization:

$$(x-3)(x^2 + x + 3)(x^3 - 2x^2 - x + 1)^2 \times (x^6 + 3x^5 + 9x^4 - x^3 - 3x^2 - 23x + 43).$$

By Theorem 2, $\mathrm{Gal}(f) \subset G_{12}$. Moreover, since $f_{10}(x)$ has an irreducible sextic factor, by Table I, we can conclude that $\mathrm{Gal}(f) = G_{12}$ or C_6. By calculating the resolvent polynomials for M, N in (2), each of which has a unique rational root, we find that $\hat{M} = -28$, $\hat{N} = 0$. Hence the polynomial $g(x)$ defined in (3) is $g(x) = x^2 + 28x = x(x+28)$ and $\{\hat{\delta}^2, \hat{\mu}^2\}$ is $\{0, -28\}$, the set of roots of g.

Similarly, using Proposition 3, one can calculate that for the δ_{ij} defined in (4) we have $\hat{\delta}_{11} = \hat{\delta}_{22} = -7$, $\hat{\delta}_{33} = \hat{\delta}_{13} = \hat{\delta}_{23} = 0$ and $\hat{\delta}_{12} = 7$. Similarly, one finds $\hat{\gamma}_1 = \hat{\gamma}_2 = -1$, $\hat{\gamma}_3 = 2$. Since $\hat{\delta}_{11} \neq 0$, we can choose $j = 1$ for the definition of M_1, N_1 in (5). Then from the resolvent polynomials, one finds that $\hat{M}_1 = 42$ and $\hat{N}_1 = -343$. Thus the polynomial $g_1(x)$ defined in (6) is $x^2 - 42x - 343 = (x-49)(x+7)$ with roots -7, 49. Hence $\{\hat{\delta}_1^2, \hat{\mu}_1^2\} = \{-7, 49\}$. Thus by Theorem 7, $\mathrm{Gal}(f)$ is either C_6 or H_6. Combining this with what was found above, we thus know that $\mathrm{Gal}(f) = C_6$.

We now find the roots of f. Following the algorithm in Fig. 3, we have $v = v_1 = -7$, $v_2 = 7$, and $v_3 = 0$.

Now $\hat{\gamma}_1 = -1$, and by calculating $\hat{\gamma}_2$, $\hat{\gamma}_3$ as roots of their respective resolvent polynomials, we find $\hat{\gamma}_2 = -1$, $\hat{\gamma}_3 = 2$. Following Fig. 3, we have

$$f_\varepsilon(x) = x^3 + \frac{1 - i\varepsilon\sqrt{7}}{2}x^2 - \frac{1 + i\varepsilon\sqrt{7}}{2}x - 1.$$

The roots $z_\varepsilon(1)$, $z_\varepsilon(2)$, $z_\varepsilon(3)$ of the equation $f_\varepsilon(x) = 0$ are then given by

$$\frac{1 - i\varepsilon\sqrt{7}}{6} + d_1 + d_2, \qquad \frac{1 - i\varepsilon\sqrt{7}}{6} + d_1\omega + d_2\omega^2,$$
$$\frac{1 - i\varepsilon\sqrt{7}}{6} + d_1\omega^2 + d_2\omega,$$

where

$$d_1 = \frac{1}{6}\sqrt[3]{56 - 4\varepsilon\sqrt{7}i + 12\sqrt{21}}, \qquad d_2 = -i\varepsilon\sqrt{7}/9d_1,$$

and $\omega = e^{2\pi i/3}$ is a primitive cubic root of unity. Then the $z_\varepsilon(i)$ for $\varepsilon = \pm 1$, $i = 1, 2, 3$ are the primitive 7th roots of unity.

Finally, we demonstrate how we can renumber the roots so that $\tau \in \mathrm{Gal}(f) = C_6$ acts on the z_i by $\sigma(z_i) = z_{\sigma(i)}$. We first let $z_{2i-1} = z_1(i)$ and $z_{2i} = z_{-1}(i)$ for $i = 1, 2, 3$. Define $n_{\sigma,0}$ as in Step 2 of Fig. 5. There is a unique choice $\sigma = e \in H$ such that $n_{\sigma,0}$ is integral and occurs with multiplicity one in the set $\{n_{\sigma,0}\}$. Hence, following Step 3 of Fig. 5, with the above numbering, $\tau \in \mathrm{Gal}(f) = C_6$ acts on the z_i by $\tau(z_i) = z_{\tau(i)}$.

Given a cubic polynomial $f(x) = x^3 + a_1x^2 + a_2x + a_3$, define

$$c_1 = \frac{1}{6}(a_1a_2 - 3a_3) - \frac{1}{27}a_1^3,$$
$$c_2 = \frac{1}{3}a_2 - \frac{1}{9}a_1^2,$$
$$d_1 = \sqrt[3]{c_1 + \sqrt{c_2^3 + c_1^2}},$$
$$d_2 = -c_2/d_1.$$

Letting $\overline{\alpha}$ denote the complex conjugate of α, define

$$y_f(\alpha) = -\frac{a_1}{3} + d_1\alpha + d_2\overline{\alpha}.$$

The roots of f are then $y_f(1)$, $y_f(\omega)$, and $y_f(\omega^2)$, where ω is a primitive cubic root of unity.

FIG. 6. Formulas for the roots of a cubic polynomial $f(x) = x^3 + a_1x^2 + a_2x + a_3$.

One can easily check that the original z_i found were $z_1 = \omega_7^5$, $z_2 = \omega_7^2$, $z_3 = \omega_7^3$, $z_4 = \omega_7^4$, $z_5 = \omega_7^6$, and $z_6 = \omega_7$. The cycle $\sigma = (145236)$ is a generator of C_6. It sends z_6 to z_1 and is the order 6 automorphism of $\mathbf{Q}[\omega_7]$ defined by $\omega_7 \mapsto \omega_7^5$. Its inverse σ^{-1} gives the map $\omega_7 \mapsto \omega_7^3$. Hence we have recovered the "classical" generator for Gal(f).

EXAMPLE 2. We now consider a less elementary example. Let $f(x)$ be the irreducible sextic polynomial $x^6 + x^4 - x^3 - 2x^2 + 3x - 1$. One can calculate that $f_{10}(x)$ factors into irreducible polynomials as

$$(x^4 - 2x^3 - x^2 + 71x + 1)$$
$$\times(x^6 - 4x^5 + 20x^4 - 30x^3 + 60x^2 - 15x + 1).$$

The resolvent $f_{15}(x)$ has the $\theta_1 = \hat{\theta}_{48} = 0$ as its unique rational root. Its factorization into irreducible polynomials is given by

$$x(x^6 - x^5 + 4x^4 + 19x^3 - 46x^2 - 82x - 31)$$
$$\times(x^8 - 2x^7 + 9x^6 - 4x^5 - 25x^4 + 53x^3 - 144x^2 - 74x + 877).$$

By Theorem 2, Gal(f) $\subset G_{48}$ and Gal(f) $\not\subset G_{72}$. Looking at the possible factorizations of $F_{15}(x)$ (and thus f_{15}) in Table I, we can further conclude that Gal(f) $= G_{48}$, Γ_{24}, or G_{24}. Using the formula in the Appendix, one calculates that $\Delta = 66309 = 69(31)^2$. Similarly, one can calculate the values $\hat{\chi}_1^2 = \hat{\chi}_2^2 = -31$, where χ_i is defined following (8). We thus let $\chi = -31$ and by Theorem 3, we conclude that Gal(f) $= G_{48}$.

To find formulas for the roots of f, by the algorithm listed in Fig. 4, we must calculate $\hat{D}$, $\hat{E}$ as roots of their G_{48}-resolvents. We find $\hat{D} = -1$,

$\hat{E} = 1$. Following the definitions in (10), (11), we have

$$g_2(x) = g_3(x) = x^3 + x + 1.$$

Letting $y_g(\omega^i)$ be defined as above, we let $l_i = m_i = y_{g_2}(\omega^i)$. We have

$$l_1 = m_1 = d_1\omega + d_2\omega^2,$$
$$l_2 = m_2 = d_1\omega^2 + d_2\omega,$$
$$l_3 = m_3 = d_1 + d_2,$$

where

$$d_1 = \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{31}{108}}}, \qquad d_2 = -1/3d_1.$$

Define $p_{k\sigma}$, $k = 1, 2$, $\sigma \in S_3$ as in Step 6 of Fig. 4. Calculation shows $p_{1(1)} = -2$, $p_{1(123)} = p_{1(132)} = 1$, and $p_{2(1)} = -3$ is the only integral value amongst the $p_{2\sigma}$. Hence, without needing to calculate $\hat{h}_{11}$, $\hat{h}_{12}$ (which equal -2, -3, respectively), by Step 7 of the algorithm, we can determine that $\sigma = 1$. Then for each $i = 1, 2, 3$, let z_{2i-1}, z_{2i} be the two roots of the polynomial $x^2 - l_i x + m_i$. The z_i are the roots of f and the Galois action of $\tau \in G_{48}$ on the z_i is given by $\tau(z_i) = z_{\tau(i)}$.

8. COMPUTATIONS

We now discuss the computation aspects raised in this paper. The essence of our algorithm for calculating the roots of an irreducible sextic $f(x) \in \mathbf{Q}[x]$ and determining $\mathrm{Gal}(f)$ can be summarized as follows (see Figs. 3, 4, 5 for specifics):

(1) Use coefficients a_i of f to define the discriminant Δ and the resolvent polynomials $f_{10}(x)$, $f_{15}(x)$.

(2) Use the rational roots of $f_{10}(x)$, $f_{15}(x)$, Δ to partially determine the Galois group and to define new resolvent polynomials (e.g., the Galois resolvents for γ_i, δ_{ij}, D, E, h_{11}, h_{12}) and other polynomials (e.g., g).

(3) Use the roots of these resolvents to define other polynomials (e.g., $g_1(x)$) and Galois resolvents (e.g., $g_2(x)$, $g_3(x)$).

(4) Repeat step (3) until enough polynomials are defined so that $\mathrm{Gal}(f)$ and the roots of $f(x)$ can be calculated.

The main computational aspect is to calculate the Galois resolvent for a G-polynomial $\theta \in \mathbf{Q}[x_1, \dots, x_6][x]$, when G is one of the groups $G = A_6$, G_{72}, or G_{48}. Let $F_\theta(x) = \sum_{i=0}^{n} d_i x^i \in \mathbf{Q}[x_1, \dots, x_6][x]$ denote the Galois

resolvent of θ. The task of computing F_θ consists of two steps:

(a) Determine the symmetric polynomials $d_i \in \mathbf{Q}[x_1, \dots, x_6]$.

(b) Given a S_n-invariant polynomial $f \in \mathbf{Q}[x_1, \dots, x_6]$, find a polynomial $\tilde{f} \in \mathbf{Q}[z_1, \dots, z_6]$ such that

$$f(x_1, \dots, x_6) = \tilde{f}(s_1, \dots, s_6). \tag{29}$$

Problem (a) is straightforward, though with θ of high degree and $G = G_{48}$, there are some concerns regarding memory and speed. For example, the coefficient c_{10} of $f_{10}(x)$ requires on the order of 9^{10} arithmetic operations to be expressed. All calculations for this paper were done using Mathematica on a Sun 5.6 and this step took at most one day for any of the polynomials calculated.

Solving Problem (b) was more difficult. The author used a fairly straightforward algorithm in symbolic computation to convert f from a polynomial in the x_i into a polynomial in the s_i. The only trick was writing an algorithm that minimized storage requirements as the simplest implementation of the algorithm requires vast amounts of storage as the intermediate step terms have many more non-zero coefficients than either the size of f or $\tilde{f}$ would suggest. The accuracy of the algorithm was double-checked at the end by checking that the original polynomial f was obtained when $\tilde{f}$ was expanded as a polynomial in the x_i. Since the initial draft of this paper, the author has learned that many superior algorithms are known for converting a symmetric polynomial into a polynomial in the symmetric functions s_i. We refer the reader to [3, 13] for the details.

Finally, we would like to address a natural computational question about reducing the number of independent calculations in our algorithm. Assume that $\mathrm{Gal}(f) \subset G_{48}$ and that $\hat{\theta}_{48}$ is known. Let θ be another G_{48}-polynomial. In order to calculate $\hat{\theta}$, our algorithm currently requires calculating the resolvent polynomial for θ and then determining $\hat{\theta}$ by inspection as a rational root of $\hat{F}_\theta(x)$. A natural question is that since $\theta \in \mathbf{Q}(s_1, \dots, s_6, \theta_{48})$, one should be able to find formulas for θ in terms of θ_{48}. Then the calculation of $\hat{\theta}_{48}$ would determine $\hat{\theta}$. In fact, this is the approach that Dummit [6] takes when finding formulas for the quintic. However, there are two important differences that make this a more difficult question for the sextic. The first is the size of the calculations. Dummit's solution involves Cramer's rule and uses 6×6 matrices. The equivalent computation here would require 15×15 matrices (or 10×10 for the group G_{72}). It is currently not known how to evaluate a determinant of this size symbolically. The second reason is that when one uses Cramer's rule for the sextic, the term in the denominator will be the

discriminant of $f_{10}(x)$ or $f_{15}(x)$. In the case of the quintic, the denominator is the discriminant of a sextic polynomial with no repeated roots. However, $f_{10}(x)$ and $f_{15}(x)$ will have repeated roots in some cases. Hence, the formulas one obtains by this approach may in principle be rational functions and thus not always defined. One might hope that with a different choice of polynomials $f_{10}(x)$, $f_{15}(x)$, one could get around this difficulty, but no such polynomials have yet been found.

APPENDIX

$$
\begin{aligned}
\Delta = {} & 108a_4^3a_3^4a_6 - 27a_4^2a_3^4a_5^2 - 3750a_5^5a_2a_3 - 1350a_6a_3^3a_5^3 - 22500a_6a_5^4a_4 \\
& + 320a_6a_1^4a_5^4 + 1500a_6a_5^4a_2^2 - 8748a_3^4a_6^3 + 34992a_3^2a_6^4 - 13824a_4^3a_6^3 \\
& - 13824a_2^3a_6^4 + 256a_1^5a_5^5 - 4860a_4a_2a_3^4a_6^2 - 630a_4a_2a_3^3a_5^3 \\
& + 3888a_4a_2a_3^2a_6^3 - 192a_4a_2a_1^4a_5^4 + 16a_4^4a_2^3a_1^2a_6 \\
& + 8208a_4^2a_2^2a_3^2a_6^2 - 6a_4^2a_2^2a_1^3a_5^3 + 560a_4^2a_2^2a_5^3a_3 + 4816a_4^3a_2^2a_1^2a_6^2 \\
& + 24a_4^2a_2^3a_1a_5^3 + 4816a_4^2a_2^3a_6a_5^2 - 4a_4^3a_2^3a_1^2a_5^2 - 6480a_4^2a_2a_1^2a_6^3 \\
& - 6480a_4a_2^2a_6^2a_5^2 + 1020a_4a_2^2a_1^2a_5^4 - 64a_4^4a_2^4a_6 - 4352a_4^3a_2^3a_6^2 \\
& + 16a_4^3a_2^4a_5^2 - 17280a_4^2a_2^2a_6^3 + 62208a_4a_2a_6^4 + 512a_4^5a_2^2a_6 \\
& - 128a_4^4a_2^2a_5^2 + 512a_4^2a_2^5a_6^2 - 900a_4a_2^3a_5^4 + 2000a_4^2a_2a_5^4 \\
& + 9216a_4^4a_2a_6^2 + 9216a_4a_2^4a_6^3 + 1500a_4^2a_1^4a_6^3 - 32400a_4a_1^2a_6^4 \\
& - 36a_4^3a_1^3a_5^3 + 108a_4^5a_1^4a_6 - 27a_4^4a_1^4a_5^2 - 50a_4^2a_1^2a_5^4 - 192a_4^4a_1^2a_6^2 \\
& + 27000a_6^2a_5^3a_3 - 1350a_3^3a_1^3a_6^3 + 38880a_6^4a_1a_5 + 540a_6^3a_1^2a_5^2 \\
& - 32400a_6^3a_5^2a_2 + 27000a_3a_1^3a_6^4 + 410a_6^2a_1^3a_5^3 - 8640a_3^2a_2^3a_6^3 \\
& + 43200a_1^2a_2^2a_6^4 + 43200a_5^2a_4^2a_6^2 - 8640a_4^3a_3^2a_6^2 - 192a_2^4a_5^2a_6^2 \\
& - 22500a_2a_1^4a_6^4 - 900a_1a_5^4a_3^3 - 128a_1^4a_5^4a_3^2 + 2000a_1^2a_5^5a_3 \\
& - 1600a_1^3a_5^5a_2 + 2250a_1a_5^5a_2^2 - 2500a_1a_5^5a_4 + 2250a_4a_3^2a_5^4 + 825a_3^2a_2^2a_5^4 \\
& - 1600a_4^3a_5^3a_3 + 19800a_4a_2a_6a_5^3a_3 - 46656a_6^5 + 2808a_4a_2^2a_3^3a_1a_6^2 \\
& + 2808a_4^2a_2a_3^3a_5a_6 - 4536a_4^2a_2a_3^2a_1^2a_6^2 - 22896a_4a_2a_3^2a_6^2a_1a_5 \\
& + 356a_4a_2^2a_3^2a_1a_5^3 - 4536a_4a_2^2a_3^2a_6a_5^2 + 144a_4a_2^3a_3^2a_1^2a_6^2 \\
& + 18a_4a_2^3a_3a_1^2a_5^3 - 3456a_4a_2^2a_3a_1a_6^3 - 13040a_4^2a_2a_1^3a_5a_6^2
\end{aligned}
$$

$$
\begin{aligned}
&-5760a_4^3a_2a_1a_3a_6^2 - 5760a_4a_2^3a_3a_5a_6^2 - 3456a_4^2a_2a_5a_3a_6^2\\
&+1020a_4^2a_2a_1^4a_5^2a_6 - 746a_4^2a_2a_1^2a_5^3a_3 - 2050a_4a_2a_1^4a_5a_3a_6^2\\
&-80a_4a_2a_3^2a_1^3a_5^3 - 630a_4a_2a_3^3a_1^3a_6^2 + 31968a_4a_2a_6^3a_1a_5\\
&+8748a_4a_2a_6^2a_1^2a_5^2 + 19800a_4a_2a_3a_1^3a_6^3 - 2050a_4a_2a_1a_5^4a_3\\
&-1584a_4^2a_2^2a_3^2a_6a_1a_5 - 2496a_4^2a_2^3a_3a_1a_6^2 + 24a_4^3a_2^2a_1^3a_5a_6\\
&+320a_4^4a_2^2a_1a_3a_6 - 80a_4^3a_2^2a_1a_5^2a_3 + 320a_4^2a_2^4a_3a_5a_6 - 2496a_4^3a_2^2a_5a_3a_6\\
&+15264a_4^2a_2^2a_6^2a_1a_5 - 5428a_4^2a_2^2a_6a_1^2a_5^2 + 560a_4^2a_2^2a_3a_1^3a_6^2\\
&-96a_4^3a_2^3a_6a_1a_5 - 80a_4^2a_2^3a_3a_1^2a_6a_5 + 356a_4^2a_2a_3^2a_1^3a_5a_6\\
&+10152a_4a_2^2a_3a_1^2a_6^2a_5 - 746a_4a_2^2a_3a_1^3a_6a_5^2 + 3272a_4a_2^3a_3a_1a_6a_5^2\\
&+3272a_4^3a_2a_1^2a_5a_3a_6 + 9768a_4a_2a_6a_1^3a_5^3 - 72a_4a_2^4a_3a_5^3\\
&-576a_4a_2^4a_3^2a_6^2 - 10560a_4a_2^3a_1^2a_6^3 + 160a_4^3a_2a_1a_5^3 - 10560a_4^3a_2a_5^2a_6\\
&-900a_4^3a_2a_1^4a_6^2 - 576a_4^5a_2a_1^2a_6 + 144a_4^4a_2a_1^2a_5^2 - 576a_4^4a_2a_3^2a_6\\
&+144a_4^3a_2a_3^2a_5^2 - 576a_4a_2^5a_5^2a_6 + 2000a_4a_2^2a_1^4a_6^3 - 128a_4^2a_2^4a_1^2a_6^2\\
&+162a_4a_1^2a_3^4a_6^2 + 24a_4a_1^2a_3^3a_5^3 - 27540a_4a_1^2a_3^2a_6^3 + 825a_4^2a_1^4a_3^2a_6^2\\
&+2250a_4^2a_1^5a_5a_6^2 - 120a_4^3a_1^3a_3a_6^2 + 144a_4^2a_1^4a_5^3a_3 - 1800a_4a_1^3a_6^3a_5\\
&-1700a_4a_1^4a_6^2a_5^2 - 3750a_4a_1^5a_3a_6^3 + 160a_4a_1^3a_5^4a_3 - 1600a_4a_1^5a_6a_5^3\\
&+248a_4^3a_1^2a_5^2a_6 + 24a_4^4a_1^2a_3^2a_6 - 6a_4^3a_1^2a_3^2a_5^2 + 144a_4^4a_1^3a_5a_6\\
&+21384a_3^3a_1a_2a_6^3 + 21384a_3^3a_5a_4a_6^2 + 15552a_3^2a_6^3a_1a_5\\
&-27540a_3^2a_6^2a_5^2a_2 - 9720a_3^2a_1^2a_2^2a_6^3 - 77760a_3a_1a_2a_6^4\\
&+46656a_1a_4^2a_3a_6^3 + 46656a_3a_2^2a_5a_6^3 - 77760a_5a_4a_3a_6^3\\
&+2250a_1^4a_5a_3a_6^3 - 1800a_6^2a_1a_5^3a_2 + 248a_1^2a_2^3a_5^2a_6^2 - 21888a_1a_2^3a_5a_6^3\\
&+15600a_1^3a_2^2a_6^3a_5 - 21888a_5a_1a_4^3a_6^2 - 6318a_1a_5a_3^4a_6^2\\
&+15417a_1^2a_5^2a_3^2a_6^2 + 560a_1^2a_5^4a_3^2a_2 + 144a_1^3a_5^4a_3a_2^2 + 2000a_1^5a_5^2a_3a_6^2\\
&-900a_1^4a_5a_3^3a_6^2 - 630a_1a_5^4a_3a_2^3 + 1020a_1a_5^3a_4^2a_3^2 + 144a_1a_5^3a_2^4a_6\\
&+2250a_6a_1a_5^4a_3 - 1700a_6a_1^2a_5^4a_2 - 120a_6a_3a_2^3a_5^3 + 15600a_6a_1a_4^2a_5^3\\
&-9720a_6a_4^2a_3^2a_5^2 + 10152a_4^2a_2a_6a_1a_5^2a_3 - 13040a_4a_2^2a_6a_1a_5^3\\
&+144a_4a_2^4a_1^2a_5^3a_6 - 640a_4a_2^4a_1a_5a_6^2 + 160a_4a_2^3a_1^3a_6^2a_5
\end{aligned}
$$

$$
\begin{aligned}
&-72a_4^4a_2a_1^3a_3a_6 + 18a_4^3a_2a_1^3a_3a_5^2 - 640a_4^4a_2a_5a_1a_6\\
&-12330a_4a_1^2a_6a_5^3a_3 - 108a_4^2a_1^2a_3^3a_5a_6 + 1980a_4a_1^3a_3^2a_6^2a_5\\
&-2412a_4a_1^2a_3^2a_6a_5^2a_2 + 16632a_4^2a_1^2a_5a_3a_6^2 - 630a_4^3a_1^4a_5a_3a_6\\
&-682a_4^2a_1^3a_6a_5^2a_3 - 31320a_3a_1^2a_2a_6^3a_5 - 12330a_3a_1^3a_2a_6^2a_5^2\\
&+16632a_3a_1a_2^2a_6^2a_5^2 - 31320a_6^2a_1a_4a_5^2a_3 + 3942a_1^2a_5a_3^3a_2a_6^2\\
&+3942a_1a_5^2a_3^3a_4a_6 + 1020a_1^3a_5a_3^2a_2^2a_6^2 + 560a_1^4a_5^2a_3^2a_4a_6\\
&+160a_1^4a_5^3a_3a_2a_6 - 4464a_1a_5a_3^2a_2^3a_6^3 - 4464a_1a_5a_4^3a_3^2a_6\\
&+1980a_6a_3^2a_1a_5^3a_2 - 682a_6a_3a_1^2a_2^2a_5^3 + 3125a_5^6 + 16a_3^3a_2^3a_5^3\\
&+108a_3^4a_2^3a_6^2 + 16a_3^4a_1^3a_5^3 + 108a_3^5a_1^3a_6^2 - 27a_2^4a_5^4a_1^2 + 256a_2^5a_1^2a_6^3\\
&+5832a_1a_4^2a_3^3a_6^2 + 768a_1a_4^5a_3a_6 - 192a_1a_4^4a_5^2a_3 + 162a_3^4a_2a_5^2a_6\\
&+24a_3^2a_2^4a_5^2a_6 + 16a_3^2a_2^3a_4^3a_6 + 2250a_3^2a_2a_1^4a_6^3 + 6912a_3a_2^4a_1a_6^3\\
&-72a_3^4a_2a_1a_5^3 - 486a_3^5a_2a_1a_6^2 + 768a_3a_2^5a_5a_6^2 - 1600a_3a_2^3a_1^3a_6^3\\
&-4a_3^2a_2^3a_4^2a_5^2 - 27a_3^4a_2^2a_1^2a_6^2 - 4a_3^3a_2^2a_1^2a_5^3 + 16a_3^3a_1^3a_4^3a_6\\
&-4a_3^3a_1^3a_4^2a_5^2 + 5832a_5a_3^3a_2^2a_6^2 + 6912a_5a_4^4a_3a_6 - 1024a_4^6a_6\\
&+256a_4^5a_5^2 + 108a_2^5a_5^4 - 1024a_2^6a_6^3 + 108a_3^5a_5^3 + 729a_3^6a_6^2\\
&-72a_3^3a_2a_4^3a_1a_6 + 18a_3^3a_2a_4^2a_1a_5^2 - 108a_3^3a_2^2a_5^2a_6a_1 - 6a_3^2a_2^3a_5^2a_1^2a_6\\
&-4a_3^2a_2^2a_4^3a_1^2a_6 + 324a_3^4a_2a_4a_6a_1a_5 - 72a_3^3a_2^3a_4a_5a_6\\
&-192a_3a_2^4a_1^2a_5a_6^2 + 24a_3^3a_2a_1^3a_6a_5^2 + a_3^2a_2^2a_4^2a_1^2a_5^2 + 18a_3^3a_2^2a_4a_1^2a_6a_5\\
&-72a_3^4a_1^3a_4a_6a_5 - 486a_4a_3^5a_5a_6 + 3125a_1^6a_6^4
\end{aligned}
$$

$$b_1 = 6a_2$$

$$b_2 = 15a_2^2 + 3a_1a_3 - 6a_4$$

$$b_3 = 20a_2^3 + 15a_1a_2a_3 - 3a_3^2 + \left(-a_1^2 - 22a_2\right)a_4 - 11a_1a_5 + 66a_6$$

$$
\begin{aligned}
b_4 = {} & 15a_2^4 + 30a_1a_2^2a_3 + \left(3a_1^2 - 12a_2\right)a_3^2 + \left(-3a_1^2a_2 - 28a_2^2 - 13a_1a_2\right)a_4\\
&+ a_4^2 + \left(-3a_1^3 - 47a_1a_2 + 36a_3\right)a_5 + \left(58a_1^2 + 138a_2\right)a_6
\end{aligned}
$$

$$
\begin{aligned}
b_5 = {} & 6a_2^5 + 30a_1a_2^3a_3 + 12a_1^2a_2a_3^2 - 18a_2^2a_3^2 - 6a_1a_3^3 - 2a_1^2a_2^2a_4\\
&- 12a_2^3a_4 - 2a_1^3a_3a_4 - 36a_1a_2a_3a_4 + 12a_3^2a_4 - 4a_1^2a_4^2 - 12a_2a_4^2\\
&- 13a_1^3a_2a_5 - 78a_1a_2^2a_5 + 3a_1^2a_3a_5 + 96a_2a_3a_5 + 63a_1a_4a_5
\end{aligned}
$$

$$- 123a_5^2 + 11a_1^4a_6 + 156a_1^2a_2a_6 + 84a_2^2a_6 - 57a_1a_3a_6 + 114a_4a_6$$

$$\begin{aligned}
b_6 = {} & a_2^6 + 15a_1a_2^4a_3 + 18a_1^2a_2^2a_3^2 - 12a_2^3a_3^2 + a_1^3a_3^3 - 18a_1a_2a_3^3 + 3a_3^4 \\
& + 2a_1^2a_2^3a_4 + 2a_2^4a_4 - 4a_1^3a_2a_3a_4 - 30a_1a_2^2a_3a_4 - 6a_1^2a_3^2a_4 \\
& + 20a_2a_3^2a_4 - a_1^4a_4^2 - 20a_1^2a_2a_4^2 - 26a_2^2a_4^2 + 10a_1a_3a_4^2 + 24a_4^3 \\
& - 22a_1^3a_2^2a_5 - 62a_1a_2^3a_5 - 2a_1^4a_3a_5 + 88a_2^2a_3a_5 + 46a_1a_3^2a_5 \\
& + 32a_1^3a_4a_5 + 140a_1a_2a_4a_5 - 138a_3a_4a_5 - 111a_1^2a_5^2 - 94a_2a_5^2 \\
& + 33a_1^4a_2a_6 + 156a_1^2a_2^2a_6 + 20a_2^3a_6 - 3a_1^3a_3a_6 - 228a_1a_2a_3a_6 \\
& + 138a_3^2a_6 + 113a_1^2a_4a_6 + 88a_2a_4a_6 - 43a_1a_5a_6 + 129a_6^2
\end{aligned}$$

$$\begin{aligned}
b_7 = {} & 3a_1a_2^5a_3 + 12a_1^2a_2^3a_3^2 - 3a_2^4a_3^2 + 3a_1^3a_2a_3^3 - 18a_1a_2^2a_3^3 - 3a_1^2a_3^4 \\
& + 6a_2a_3^4 + 3a_1^2a_2^4a_4 + 2a_2^5a_4 - 4a_1a_2^3a_3a_4 - a_1^4a_3^2a_4 \\
& - 14a_1^2a_2a_3^2a_4 + 4a_2^2a_3^2a_4 + 14a_1a_3^3a_4 - 4a_1^4a_2a_4^2 - 28a_1^2a_2^2a_4^2 \\
& - 12a_2^3a_4^2 - 2a_1^3a_3a_4^2 + 6a_1a_2a_3a_4^2 - 2a_3^2a_4^2 + 30a_1^2a_4^3 + 16a_2a_4^3 \\
& - 18a_1^3a_2^3a_5 - 23a_1a_2^4a_5 - 8a_1^4a_2a_3a_5 - 10a_1^2a_2^2a_3a_5 + 32a_2^3a_3a_5 \\
& + 17a_1^3a_3^2a_5 + 82a_1a_2a_3^2a_5 - 36a_3^3a_5 + 4a_1^5a_4a_5 + 72a_1^3a_2a_4a_5 \\
& + 90a_1a_2^2a_4a_5 - 72a_1^2a_3a_4a_5 - 82a_2a_3a_4a_5 - 76a_1a_4^2a_5 \\
& - 36a_1^4a_5^2 - 76a_1^2a_2a_5^2 - 44a_2^2a_5^2 + 10a_1a_3a_5^2 + 94a_4a_5^2 \\
& + 38a_1^4a_2^2a_6 + 80a_1^2a_2^3a_6 + 10a_2^4a_6 + 4a_1^5a_3a_6 - 60a_1^3a_2a_3a_6 \\
& - 230a_1a_2^2a_3a_6 + 52a_1^2a_3^2a_6 + 186a_2a_3^2a_6 + 48a_1^4a_4a_6 \\
& + 76a_1^2a_2a_4a_6 - 36a_2^2a_4a_6 + 34a_1a_3a_4a_6 + 80a_4^2a_6 - 88a_1^3a_5a_6 \\
& + 184a_1a_2a_5a_6 - 342a_3a_5a_6 + 74a_1^2a_6^2 + 132a_2a_6^2
\end{aligned}$$

$$\begin{aligned}
b_8 = {} & 3a_1^2a_2^4a_3^2 + 3a_1^3a_2^2a_3^3 - 6a_1a_2^3a_3^3 - 6a_1^2a_2a_3^4 + 3a_2^2a_3^4 + 3a_1a_3^5 \\
& + a_1^2a_2^5a_4 + 4a_1^3a_2^3a_3a_4 + 3a_1a_2^4a_3a_4 - a_1^4a_2a_3^2a_4 - 10a_1^2a_2^2a_3^2a_4 \\
& - 4a_2^3a_3^2a_4 + a_1^3a_3^3a_4 + 12a_1a_2a_3^3a_4 - 6a_3^4a_4 - 5a_1^4a_2^2a_4^2 - 12a_1^2a_2^3a_4^2 \\
& + a_2^4a_4^2 - a_1^5a_3a_4^2 - 12a_1^3a_2a_3a_4^2 - 2a_1a_2^2a_3a_4^2 + 13a_1^2a_3^2a_4^2 \\
& + 12a_2a_3^2a_4^2 + 10a_1^4a_4^3 + 28a_1^2a_2a_4^3 - 8a_2^2a_4^3 - 24a_1a_3a_4^3 + 16a_4^4 \\
& - 7a_1^3a_2^4a_5 - 3a_1a_2^5a_5 - 11a_1^4a_2^2a_3a_5 - 8a_1^2a_2^3a_3a_5 + 4a_2^4a_3a_5 \\
& + a_1^5a_3^2a_5 + 29a_1^3a_2a_3^2a_5 + 42a_1a_2^2a_3^2a_5 - 15a_1^2a_3^3a_5 - 24a_2a_3^3a_5
\end{aligned}$$

$$+ 9a_1^5a_2a_4a_5 + 48a_1^3a_2^2a_4a_5 + 12a_1a_2^3a_4a_5 - 9a_1^4a_3a_4a_5$$
$$- 33a_1^2a_2a_3a_4a_5 - 6a_2^2a_3a_4a_5 - 9a_1a_3^2a_4a_5$$

$$\begin{aligned}
b_9 = {} & a_1^3a_2^3a_3^3 - 3a_1^2a_2^2a_3^4 + 3a_1a_2a_3^5 - a_3^6 + 2a_1^3a_2^4a_3a_4 + a_1^4a_2^2a_3^2a_4 \\
& -2a_1^2a_2^3a_3^2a_4 - 2a_1^3a_2a_3^3a_4 - 2a_1a_2^2a_3^3a_4 + a_1^2a_3^4a_4 + 2a_2a_3^4a_4 \\
& -2a_1^4a_2^3a_4^2 - 3a_1^5a_2a_3a_4^2 - 10a_1^3a_2^2a_3a_4^2 + 2a_1a_2^3a_3a_4^2 + 3a_1^4a_3^2a_4^2 \\
& +22a_1^2a_2a_3^2a_4^2 - 2a_2^2a_3^2a_4^2 - 10a_1a_3^3a_4^2 + a_1^6a_4^3 + 10a_1^4a_2a_4^3 \\
& -2a_1^2a_2^2a_4^3 - 10a_1^3a_3a_4^3 - 8a_1a_2a_3a_4^3 + 8a_3^2a_4^3 + 8a_1^2a_4^4 - a_1^3a_2^5a_5 \\
& -6a_1^4a_2^3a_3a_5 - a_1^2a_2^4a_3a_5 + a_1^5a_2a_3^2a_5 + 15a_1^3a_2^2a_3^2a_5 + 6a_1a_2^3a_3^2a_5 \\
& -a_1^4a_3^3a_5 - 10a_1^2a_2a_3^3a_5 - 4a_2^2a_3^3a_5 + a_1a_3^4a_5 + 6a_1^5a_2^2a_4a_5 \\
& +8a_1^3a_2^3a_4a_5 - a_1a_2^4a_4a_5 + 2a_1^4a_2a_3a_4a_5 - 10a_1^2a_2^2a_3a_4a_5 \\
& +2a_2^3a_3a_4a_5 - 7a_1^3a_3^2a_4a_5 + 10a_1a_2a_3^2a_4a_5 - 6a_3^3a_4a_5 - 13a_1^5a_4^2a_5 \\
& -6a_1^3a_2a_4^2a_5 + 4a_1a_2^2a_4^2a_5 - 10a_1^2a_3a_4^2a_5 - 8a_2a_3a_4^2a_5 \\
& -4a_1^6a_2a_5^3 - 4a_1^4a_2^2a_5^2 + 2a_1^2a_2^3a_5^2 - a_2^4a_5^2 + 4a_1^5a_3a_5^2 \\
& -11a_1^3a_2a_3a_5^2 - 20a_1a_2^2a_3a_5^2 + 14a_1^2a_3^2a_5^2 + 16a_2a_3^2a_5^2 + 19a_1^4a_4a_5^2 \\
& +6a_1^2a_2a_4a_5^2 + 6a_2^2a_4a_5^2 + 18a_1a_3a_4a_5^2 - 8a_4^2a_5^2 - 7a_1^3a_5^3 \\
& +6a_1a_2a_5^3 - 14a_3a_5^3 + 7a_1^4a_2^4a_6 + 4a_1^2a_2^5a_6 - 2a_1^5a_2^2a_3a_6 \\
& -28a_1^3a_2^3a_3a_6 - 9a_1a_2^4a_3a_6 + a_1^6a_3^2a_6 + 42a_1^2a_2^2a_3^2a_6 + 6a_2^3a_3^2a_6 \\
& +5a_1^3a_3^3a_6 - 30a_1a_2a_3^3a_6 + 12a_3^4a_6 + 10a_1^6a_2a_4a_6 - 4a_1^4a_2^2a_4a_6 \\
& -20a_1^2a_2^3a_4a_6 + 2a_2^4a_4a_6 - 2a_1^5a_3a_4a_6 - 10a_1^3a_2a_3a_4a_6 \\
& +30a_1a_2^2a_3a_4a_6 + 26a_1^2a_3^2a_4a_6 - 4a_2a_3^2a_4a_6 + 14a_1^4a_4^2a_6 \\
& +36a_1^2a_2a_4^2a_6 - 16a_2^2a_4^2a_6 - 56a_1a_3a_4^2a_6 + 32a_4^3a_6 - 4a_1^7a_5a_6 \\
& -6a_1^5a_2a_5a_6 + 44a_1^3a_2^2a_5a_6 + 16a_1a_2^3a_5a_6 - 14a_1^4a_3a_5a_6 \\
& -74a_1^2a_2a_3a_5a_6 - 26a_2^2a_3a_5a_6 + 4a_1a_3^2a_5a_6 - 36a_1^3a_4a_5a_6 \\
& -24a_1a_2a_4a_5a_6 + 24a_3a_4a_5a_6 + 30a_1^2a_5^2a_6 + 20a_2a_5^2a_6 - 6a_1^6a_6^2 \\
& +48a_1^4a_2a_6^2 - 26a_1^2a_2^2a_6^2 + 4a_2^3a_6^2 - 52a_1^3a_3a_6^2 + 72a_1a_2a_3a_6^2 \\
& +48a_3^2a_6^2 + 72a_1^2a_4a_6^2 - 16a_2a_4a_6^2 - 32a_1a_5a_6^2 + 64a_6^3
\end{aligned}$$

$$b_{10} = a_1^4a_2^3a_3^2a_4 - 3a_1^3a_2^2a_3^3a_4 + 3a_1^2a_2a_3^4a_4 - a_1a_3^5a_4 - 2a_1^5a_2^2a_3a_4^2$$

$$+ 4a_1^4a_2a_3^2a_4^2 + a_1^2a_2^2a_3^2a_4^2 - 2a_1^3a_3^3a_4^2 - 2a_1a_2a_3^3a_4^2 + a_3^4a_4^2 + a_1^6a_2a_4^3$$
$$- a_1^5a_3a_4^3 - 2a_1^3a_2a_3a_4^3 + 2a_1^2a_3^2a_4^3 + a_1^4a_4^4 - a_1^4a_2^4a_3a_5 + 3a_1^3a_2^3a_3^2a_5$$
$$- 3a_1^2a_2^2a_3^3a_5 + a_1a_2a_3^4a_5 + a_1^5a_2^3a_4a_5 + a_1^6a_2a_3a_4a_5 - a_1^4a_2^2a_3a_4a_5$$
$$- a_1^2a_2^3a_3a_4a_5 - a_1^5a_3^2a_4a_5 + a_1^3a_2a_3^2a_4a_5 + 3a_1a_2^2a_3^2a_4a_5 - a_1^2a_3^3a_4a_5$$
$$- 2a_2a_3^3a_4a_5 - a_1^7a_4^2a_5 - a_1^5a_2a_4^2a_5 + a_1^3a_2^2a_4^2a_5 - a_1^4a_3a_4^2a_5$$
$$- 2a_1^2a_2a_3a_4^2a_5 - a_1^6a_2^2a_5^2 + a_1^4a_2^3a_5^2 + a_1^5a_2a_3a_5^2 - 5a_1^3a_2^2a_3a_5^2$$
$$- a_1a_2^3a_3a_5^2 + 5a_1^2a_2a_3^2a_5^2 + a_2^2a_3^2a_5^2 - a_1a_3^3a_5^2 + 2a_1^6a_4a_5^2$$
$$- a_1^4a_2a_4a_5^2 + 2a_1^2a_2^2a_4a_5^2 + 5a_1^3a_3a_4a_5^2 - 2a_1a_2a_3a_4a_5^2 + 2a_3^2a_4a_5^2$$
$$- 2a_1^2a_4^2a_5^2 - a_1^5a_5^3 + a_1^3a_2a_5^3 + a_1a_2^2a_5^3 - 3a_1^2a_3a_5^3 - 2a_2a_3a_5^3$$
$$+ a_5^4 + a_1^4a_2^5a_6 - a_1^5a_2^3a_3a_6 - 3a_1^3a_2^4a_3a_6 + 3a_1^4a_2^2a_3^2a_6$$
$$+ 3a_1^2a_2^3a_3^2a_6 - 3a_1^3a_2a_5^3a_6 - a_1a_2^2a_3^3a_6 + a_1^2a_3^4a_6 + 3a_1^6a_2^2a_4a_6$$
$$- 6a_1^4a_2^3a_4a_6 + a_1^2a_2^4a_4a_6 + a_1^7a_3a_4a_6 - 10a_1^5a_2a_3a_4a_6$$
$$+ 15a_1^3a_2^2a_3a_4a_6 - 2a_1a_2^3a_3a_4a_6 + 10a_1^4a_3^2a_4a_6 - 14a_1^2a_2a_3^2a_4a_6$$
$$+ 2a_2^2a_3^2a_4a_6 + 8a_1a_3^3a_4a_6 + 10a_1^4a_2a_4^2a_6 - 6a_1^2a_2^2a_4^2a_6$$
$$- 16a_1^3a_3a_4^2a_6 + 8a_1a_2a_3a_4^2a_6$$

$$c_1 = 3a_2$$
$$c_2 = 3a_2^2 + 3a_1a_3 - 6a_4$$
$$c_3 = a_2^3 + 6a_1a_2a_3 + 3a_3^2 + 3a_1^2a_4 - 20a_2a_4 - 7a_1a_5 + 42a_6$$
$$c_4 = 3a_1a_2^2a_3 + \left(3a_1^2 + 6a_2\right)a_3^2 + \left(6a_1^2 - 22a_2\right)a_2a_4 - 10a_1a_3a_4 + 7a_4^2$$
$$+\left(3a_1^3 - 26a_1a_2 + 9a_3\right)a_5 + \left(-8a_1^2 + 120a_2\right)a_6$$
$$c_5 = \left(3a_1^2 + 3a_2\right)a_2a_3^2 + 6a_1a_3^3 + \left(3a_1^2 - 8a_2\right)a_2^2a_4$$
$$+\left(6a_1^3 - 24a_1a_2 - 12a_3\right)a_3a_4 + \left(-12a_1^2 + 35a_2\right)a_4^2 + 6a_1^3a_2a_5$$
$$- 31a_1a_2^2a_5 + \left(-13a_1^2 + 15a_2\right)a_3a_5 + 23a_1a_4a_5 + 21a_5^2$$
$$+\left(3a_1^4 + -32a_1^2a_2 + 126a_2^2 + 111a_1a_3 - 222a_4\right)a_6$$
$$c_6 = a_1^3a_3^3 + 6a_1a_2a_3^3 + 3a_3^4 + 6a_1^3a_2a_3a_4 - 14a_1a_2^2a_3a_4 + 4a_1^2a_5^2a_4$$
$$- 28a_2a_3^2a_4 + 3a_1^4a_4^2 - 28a_1^2a_2a_4^2 + 50a_2^2a_4^2 - 5a_1a_3a_4^2 + 8a_4^3$$
$$+ 3a_1^3a_2^2a_5 - 12a_1a_2^3a_5 + 6a_1^4a_3a_5 - 36a_1^2a_2a_3a_5 + 4a_2^2a_3a_5$$

$$+ a_1a_3^2a_5 - 26a_1^3a_4a_5 + 103a_1a_2a_4a_5 - 21a_3a_4a_5 + 23a_1^2a_5^2 + 17a_2a_5^2$$
$$+ 6a_1^4a_2a_6 - 40a_1^2a_2^2a_6 + 56a_2^3a_6 - 16a_1^3a_3a_6 + 225a_1a_2a_3a_6$$
$$+ 57a_3^2a_6 + 91a_1^2a_4a_6 - 602a_2a_4a_6 - 151a_1a_5a_6 + 453a_6^2$$

$$c_7 = 3a_1^2a_3^4 + 3a_2a_3^4 + 3a_1^4a_3^2a_4 + 2a_1^2a_2a_3^2a_4 - 16a_2^2a_3^2a_4 - 8a_1a_3^3a_4$$
$$+ 3a_1^4a_2a_4^2 - 16a_1^2a_2^2a_4^2 + 22a_2^3a_4^2 - 8a_1^3a_3a_4^2 + 6a_1a_2a_3a_4^2$$
$$+ 5a_3^2a_4^2 + 5a_1^2a_4^3 - 4a_2a_4^3 + 6a_1^4a_2a_3a_5 - 23a_1^2a_2^2a_3a_5 - 2a_2^3a_3a_5$$
$$+ a_1^3a_3^2a_5 - 28a_1a_2a_3^2a_5 + 9a_3^3a_5 + 6a_1^5a_4a_5 - 66a_1^3a_2a_4a_5$$
$$+ 144a_1a_2^2a_4a_5 + 12a_1^2a_3a_4a_5 - 20a_2a_3a_4a_5 + 16a_1a_4^2a_5$$
$$- 14a_1^4a_5^2 + 91a_1^2a_2a_5^2 - 49a_2^2a_5^2 + 20a_1a_3a_5^2 - 109a_4a_5^2$$
$$+ 3a_1^4a_2^2a_6 - 16a_1^2a_2^3a_6 + 8a_2^4a_6 + 6a_1^5a_3a_6 - 48a_1^3a_2a_3a_6$$
$$+ 134a_1a_2^2a_3a_6 + 95a_1^2a_3^2a_6 + 120a_2a_3^2a_6 - 28a_1^4a_4a_6$$
$$+ 263a_1^2a_2a_4a_6 - 588a_2^2a_4a_6 - 409a_1a_3a_4a_6 + 340a_4^2a_6$$
$$+ 88a_1^3a_5a_6 - 529a_1a_2a_5a_6 + 207a_3a_5a_6 - 149a_1^2a_6^2 + 1173a_2a_6^2$$

$$c_8 = 3a_1a_3^5 + 8a_1^3a_3^3a_4 - 12a_1a_2a_3^3a_4 - 6a_3^4a_4 + 3a_1^5a_3a_4^2 - 12a_1^3a_2a_3a_4^2$$
$$+ 12a_1a_2^2a_3a_4^2 - 23a_1^2a_3^2a_4^2 + 28a_2a_3^2a_4^2 - 6a_1^4a_4^3 + 28a_1^2a_2a_4^3$$
$$- 40a_2^2a_4^3 + 20a_1a_3a_4^3 - 17a_4^4 + 3a_1^5a_3^2a_5 - 4a_1^3a_2a_3^2a_5 - 28a_1a_2^2a_3^2a_5$$
$$- 5a_1^2a_3^3a_5 + 6a_1^5a_2a_4a_5 - 40a_1^3a_2^2a_4a_5 + 64a_1a_2^3a_4a_5 - 20a_1^4a_3a_4a_5$$
$$+ 38a_1^2a_2a_3a_4a_5 + 32a_2^2a_3a_4a_5 - 2a_1a_3^2a_4a_5 + 22a_1^3a_4^2a_5$$
$$- 24a_1a_2a_4^2a_5 + 42a_3a_4^2a_5 + 3a_1^6a_5^2 - 38a_1^4a_2a_5^2 + 122a_1^2a_2^2a_5^2$$
$$- 72a_2^3a_5^2 + 27a_1^3a_3a_5^2 - 2a_1a_2a_3a_5^2 + 9a_3^2a_5^2 + 18a_1^2a_4a_5^2$$
$$- 236a_2a_4a_5^2 - 48a_1a_5^3 + 6a_1^5a_2a_3a_6 - 32a_1^3a_2^2a_3a_6 + 16a_1a_2^3a_3a_6$$
$$- 2a_1^4a_3^2a_6 + 74a_1^2a_2a_3^2a_6 + 72a_2^2a_3^2a_6 + 138a_1a_3^3a_6 + 6a_1^6a_4a_6$$
$$- 76a_1^4a_2a_4a_6 + 272a_1^2a_2^2a_4a_6 - 224a_2^3a_4a_6 + 146a_1^3a_3a_4a_6$$
$$- 768a_1a_2a_3a_4a_6 - 276a_3^2a_4a_6 - 220a_1^2a_4^2a_6 + 920a_2a_4^2a_6$$
$$- 30a_1^5a_5a_6 + 312a_1^3a_2a_5a_6 - 768a_1a_2^2a_5a_6 - 370a_1^2a_3a_5a_6$$
$$+ 648a_2a_3a_5a_6 + 380a_1a_4a_5a_6 + 288a_5^2a_6 + 22a_1^4a_6^2 - 344a_1^2a_2a_6^2$$

$$c_9 = a_3^6 + 7a_1^2a_3^4a_4 - 8a_2a_3^4a_4 + 7a_1^4a_3^2a_4^2 - 25a_1^2a_2a_3^2a_4^2 + 24a_2^2a_3^2a_4^2$$

$$-14a_1a_3^3a_4^2 + a_1^6a_4^3 - 8a_1^4a_2a_4^3 + 24a_1^2a_2^2a_4^3 - 28a_2^3a_4^3 - 14a_1^3a_3a_4^3$$
$$+30a_1a_2a_3a_4^3 + 14a_3^2a_4^3 + 14a_1^2a_4^4 - 39a_2a_4^4 + 7a_1^4a_3^3a_5 - 23a_1^2a_2a_3^3a_5$$
$$+8a_2^2a_3^3a_5 - a_1a_3^4a_5 + 6a_1^6a_3a_4a_5 - 34a_1^4a_2a_3a_4a_5 + 34a_1^2a_2^2a_3a_4a_5$$
$$+30a_2^3a_3a_4a_5 - 39a_1^3a_3^2a_4a_5 + 77a_1a_2a_3^2a_4a_5 - 21a_3^3a_4a_5 - 19a_1^5a_4^2a_5$$
$$+102a_1^3a_2a_4^2a_5 - 134a_1a_2^2a_4^2a_5 + 37a_1^2a_3a_4^2a_5 + 88a_2a_3a_4^2a_5$$
$$-46a_1a_4^3a_5 + 3a_1^6a_2a_5^2 - 24a_1^4a_2^2a_5^2 + 54a_1^2a_2^3a_5^2 - 26a_2^4a_5^2$$
$$-12a_1^5a_3a_5^2 + 58a_1^3a_2a_3a_5^2 - 27a_1a_2^2a_3a_5^2 + 42a_1^2a_3^2a_5^2 - 22a_2a_3^2a_5^2$$
$$+37a_1^4a_4a_5^2 - 68a_1^2a_2a_4a_5^2 - 128a_2^2a_4a_5^2 - 136a_1a_3a_4a_5^2 + 144a_4^2a_5^2$$
$$-35a_1^3a_5^3 + 14a_1a_2a_5^3 - 49a_3a_5^3 + 3a_1^6a_3^2a_6 - 10a_1^4a_2a_3^2a_6$$
$$-22a_1^2a_2^2a_3^2a_6 + 6a_2^3a_3^2a_6 + 29a_1^3a_3^3a_6 + 141a_1a_2a_3^3a_6 + 69a_3^4a_6$$
$$+6a_1^6a_2a_4a_6 - 48a_1^4a_2^2a_4a_6 + 96a_1^2a_2^3a_4a_6 - 16a_2^4a_4a_6$$
$$-24a_1^5a_3a_4a_6 + 210a_1^3a_2a_3a_4a_6 - 338a_1a_2^2a_3a_4a_6 - 87a_1^2a_3^2a_4a_6$$
$$-650a_2a_3^2a_4a_6 + 66a_1^4a_4^2a_6 - 540a_1^2a_2a_4^2a_6 + 964a_2^2a_4^2a_6$$
$$+368a_1a_3a_4^2a_6 - 300a_4^3a_6 + 6a_1^7a_5a_6 - 86a_1^5a_2a_5a_6 + 388a_1^3a_2^2a_5a_6$$
$$-524a_1a_2^3a_5a_6 + 148a_1^4a_3a_5a_6 - 920a_1^2a_2a_3a_5a_6 + 722a_2^2a_3a_5a_6$$
$$+77a_1a_3^2a_5a_6 - 212a_1^3a_4a_5a_6 + 904a_1a_2a_4a_5a_6 + 246a_3a_4a_5a_6$$
$$+224a_1^2a_5^2a_6 + 112a_2a_5^2a_6 - 16a_1^6a_6^2 + 146a_1^4a_2a_6^2 - 410a_1^2a_2^2a_6^2$$
$$+128a_2^3a_6^2 - 453a_1^3a_3a_6^2 + 2514a_1a_2a_3a_6^2 - 600a_3^2a_6^2 + 530a_1^2a_4a_6^2$$
$$-3428a_2a_4a_6^2 - 616a_1a_5a_6^2 + 1232a_6^3$$

$$c_{10} = 2a_1a_3^5a_4 + 5a_1^3a_3^3a_4^2 - 8a_1a_2a_3^3a_4^2 - 2a_3^4a_4^2 + 2a_1^5a_3a_4^3 - 8a_1^3a_2a_3a_4^3$$
$$+6a_1a_2^2a_3a_4^3 - 8a_1^2a_3^2a_4^3 + 6a_2a_3^2a_4^3 - 2a_1^4a_4^4 + 6a_1^2a_2a_4^4$$
$$-5a_2^2a_4^4 + a_1a_3a_4^4 + 2a_4^5 + 6a_1^3a_3^4a_5 - 20a_1a_2a_3^4a_5 + 12a_1^5a_3^2a_4a_5$$
$$-69a_1^3a_2a_3^2a_4a_5 + 93a_1a_2^2a_3^2a_4a_5 - 31a_1^2a_3^3a_4a_5 + 13a_2a_3^3a_4a_5$$
$$+3a_1^7a_4^2a_5 - 28a_1^5a_2a_4^2a_5 + 88a_1^3a_2^2a_4^2a_5 - 96a_1a_2^3a_4^2a_5$$
$$-40a_1^4a_3a_4^2a_5 + 128a_1^2a_2a_3a_4^2a_5 - 12a_2^2a_3a_4^2a_5 + 69a_1a_3^2a_4^2a_5$$
$$+51a_1^3a_4^3a_5 - 130a_1a_2a_4^3a_5 - 18a_3a_4^3a_5 + 3a_1^7a_3a_5^2 - 22a_1^5a_2a_3a_5^2$$
$$+41a_1^3a_2^2a_3a_5^2 - 7a_1a_2^3a_3a_5^2 - 13a_1^4a_3^2a_5^2 + 83a_1^2a_2a_3^2a_5^2 - 56a_2^2a_3^2a_5^2$$

$$
\begin{aligned}
&+20a_1a_3^3a_5^2 - 20a_1^6a_4a_5^2 + 140a_1^4a_2a_4a_5^2 - 251a_1^2a_2^2a_4a_5^2 \\
&+54a_2^3a_4a_5^2 + 86a_1^3a_3a_4a_5^2 - 224a_1a_2a_3a_4a_5^2 - 121a_3^2a_4a_5^2 \\
&-201a_1^2a_4^2a_5^2 + 372a_2a_4^2a_5^2 + 19a_1^5a_5^3 - 120a_1^3a_2a_5^3 \\
&+169a_1a_2^2a_5^3 - 126a_1^2a_3a_5^3 + 4a_2a_3a_5^3 + 381a_1a_4a_5^3 - 353a_5^4 \\
&+6a_1^5a_3^3a_6 - 27a_1^3a_2a_3^3a_6 - 7a_1a_2^2a_3^3a_6 + 79a_1^2a_3^4a_6 + 81a_2a_3^4a_6 \\
&+6a_1^7a_3a_4a_6 - 44a_1^5a_2a_3a_4a_6 + 64a_1^3a_2^2a_3a_4a_6 + 40a_1a_2^3a_3a_4a_6 \\
&+43a_1^4a_3^2a_4a_6 + 46a_1^2a_2a_3^2a_4a_6 - 418a_2^2a_3^2a_4a_6 - 313a_1a_3^3a_4a_6 \\
&-20a_1^6a_4^2a_6 + 172a_1^4a_2a_4^2a_6 - 456a_1^2a_2^2a_4^2a_6 + 408a_2^3a_4^2a_6 \\
&-261a_1^3a_3a_4^2a_6 + 626a_1a_2a_3a_4^2a_6 + 232a_3^2a_4^2a_6 + 260a_1^2a_4^3a_6 \\
&-636a_2a_4^3a_6 + 6a_1^7a_2a_5a_6 - 56a_1^5a_2^2a_5a_6 + 160a_1^3a_2^3a_5a_6 \\
&-128a_1a_2^4a_5a_6 - 28a_1^6a_3a_5a_6 + 282a_1^4a_2a_3a_5a_6 - 706a_1^2a_2^2a_3a_5a_6 \\
&+328a_2^3a_3a_5a_6 - 109a_1^3a_3^2a_5a_6 - 136a_1a_2a_3^2a_5a_6 + 243a_3^3a_5a_6 \\
&+128a_1^5a_4a_5a_6 - 882a_1^3a_2a_4a_5a_6 + 1316a_1a_2^2a_4a_5a_6 \\
&+718a_1^2a_3a_4a_5a_6 + 56a_2a_3a_4a_5a_6 - 870a_1a_4^2a_5a_6 - 180a_1^4a_5^2a_6 \\
&+1058a_1^2a_2a_5^2a_6 - 1030a_2^2a_5^2a_6 + 55a_1a_3a_5^2a_6
\end{aligned}
$$

$$
\begin{aligned}
c_{11} = {}& a_1^2a_3^4a_4^2 + 2a_2a_3^4a_4^2 + a_1^4a_3^2a_4^3 + 2a_1^2a_2a_3^2a_4^3 - 12a_2^2a_3^2a_4^3 + 2a_1a_3^3a_4^3 \\
&+2a_1^4a_2a_4^4 - 12a_1^2a_2^2a_4^4 + 17a_2^3a_4^4 + 2a_1^3a_3a_4^4 - 12a_1a_2a_3a_4^4 - 9a_3^2a_4^4 \\
&-9a_1^2a_4^5 + 28a_2a_4^5 + 2a_1^2a_3^5a_5 - 6a_2a_3^5a_5 + 8a_1^4a_3^3a_4a_5 \\
&-37a_1^2a_2a_3^3a_4a_5 + 38a_2^2a_3^3a_4a_5 - 25a_1a_3^4a_4a_5 + 5a_1^6a_3a_4^2a_5 \\
&-36a_1^4a_2a_3a_4^2a_5 + 79a_1^2a_2^2a_3a_4^2a_5 - 54a_2^3a_3a_4^2a_5 - 49a_1^3a_3^2a_4^2a_5 \\
&+123a_1a_2a_3^2a_4^2a_5 + 48a_3^3a_4^2a_5 - 10a_1^5a_4^3a_5 + 47a_1^3a_2a_4^3a_5 \\
&-44a_1a_2^2a_4^3a_5 + 82a_1^2a_3a_4^3a_5 - 138a_2a_3a_4^3a_5 - 11a_1a_4^4a_5 + 5a_1^6a_3^2a_5^2 \\
&-37a_1^4a_2a_3^2a_5^2 + 73a_1^2a_2^2a_3^2a_5^2 - 25a_2^3a_3^2a_5^2 + 25a_1a_2a_3^3a_5^2 + 6a_3^4a_5^2 \\
&+3a_1^8a_4a_5^2 - 32a_1^6a_2a_4a_5^2 + 119a_1^4a_2^2a_4a_5^2 - 169a_1^2a_2^3a_4a_5^2 \\
&+48a_2^4a_4a_5^2 - 34a_1^5a_3a_4a_5^2 + 206a_1^3a_2a_3a_4a_5^2 - 242a_1a_2^2a_3a_4a_5^2 \\
&+53a_1^2a_3^2a_4a_5^2 - 179a_2a_3^2a_4a_5^2 + 102a_1^4a_4^2a_5^2 - 477a_1^2a_2a_4^2a_5^2 \\
&+372a_2^2a_4^2a_5^2 - 58a_1a_3a_4^2a_5^2 - 48a_4^3a_5^2 - 7a_1^7a_5^3 + 64a_1^5a_2a_5^3
\end{aligned}
$$

$$-191a_1^3a_2^2a_5^3 + 195a_1a_2^3a_5^3 + 17a_1^4a_3a_5^3 - 148a_1^2a_2a_3a_5^3 + 125a_2^2a_3a_5^3$$
$$-181a_1a_3^2a_5^3 - 244a_1^3a_4a_5^3 + 905a_1a_2a_4a_5^3 + 378a_3a_4a_5^3 + 336a_1^2a_5^4$$
$$-1153a_2a_5^4 + 6a_1^4a_3^4a_6 - 29a_1^2a_2a_3^4a_6 + 6a_2^2a_3^4a_6 + 81a_1a_3^5a_6$$
$$+10a_1^6a_3^2a_4a_6 - 77a_1^4a_2a_3^2a_4a_6 + 143a_1^2a_2^2a_3^2a_4a_6 - 32a_2^3a_3^2a_4a_6$$
$$+132a_1^3a_3^3a_4a_6 - 309a_1a_2a_3^3a_4a_6 - 162a_3^4a_4a_6 + 3a_1^8a_4^2a_6$$
$$-32a_1^6a_2a_4^2a_6 + 108a_1^4a_2^2a_4^2a_6 - 120a_1^2a_2^3a_4^2a_6 + 24a_2^4a_4^2a_6$$
$$+26a_1^5a_3a_4^2a_6 - 141a_1^3a_2a_3a_4^2a_6 + 182a_1a_2^2a_3a_4^2a_6 - 338a_1^2a_3^2a_4^2a_6$$
$$+644a_2a_3^2a_4^2a_6 - 88a_1^4a_4^3a_6 + 524a_1^2a_2a_4^3a_6 - 672a_2^2a_4^3a_6$$
$$-20a_1a_3a_4^3a_6 + 258a_4^4a_6 + 6a_1^8a_3a_5a_6 - 54a_1^6a_2a_3a_5a_6$$
$$+144a_1^4a_2^2a_3a_5a_6 - 112a_1^2a_2^3a_3a_5a_6 + 56a_2^4a_3a_5a_6 + 42a_1^5a_3^2a_5a_6$$
$$-45a_1^3a_2a_3^2a_5a_6 - 355a_1a_2^2a_3^2a_5a_6 - 184a_1^2a_3^3a_5a_6 + 291a_2a_3^3a_5a_6$$
$$-42a_1^7a_4a_5a_6 + 414a_1^5a_2a_4a_5a_6 - 1254a_1^3a_2^2a_4a_5a_6$$
$$+1136a_1a_2^3a_4a_5a_6 - 294a_1^4a_3a_4a_5a_6 + 918a_1^2a_2a_3a_4a_5a_6$$
$$-98a_2^2a_3a_4a_5a_6 + 1088a_1a_3^2a_4a_5a_6 + 396a_1^3a_4^2a_5a_6$$
$$-1334a_1a_2a_4^2a_5a_6 - 1488a_3a_4^2a_5a_6 + 69a_1^6a_5^2a_6 - 678a_1^4a_2a_5^2a_6$$
$$+1950a_1^2a_2^2a_5^2a_6 - 1670a_2^3a_5^2a_6 + 434a_1^3a_3a_5^2a_6 - 721a_1a_2a_3a_5^2a_6$$
$$-48a_3^2a_5^2a_6 - 916a_1^2a_4a_5^2a_6 + 2282a_2a_4a_5^2a_6 - 32a_1a_5^3a_6 + 3a_1^8a_2a_6^2$$
$$-32a_1^6a_2^2a_6^2 + 112a_1^4a_2^3a_6^2 - 96a_1^2a_2^4a_6^2 - 112a_2^5a_6^2 - 16a_1^7a_3a_6^2$$
$$+168a_1^5a_2a_3a_6^2 - 592a_1^3a_2^2a_3a_6^2 + 648a_1a_2^3a_3a_6^2 - 349a_1^4a_3^2a_6^2$$
$$+1567a_1^2a_2a_3^2a_6^2 - 534a_2^2a_3^2a_6^2 - 648a_1a_3^3a_6^2 + 29a_1^6a_4a_6^2$$
$$-201a_1^4a_2a_4a_6^2 + 320a_1^2a_2^2a_4a_6^2 + 128a_2^3a_4a_6^2 + 972a_1^3a_3a_4a_6^2$$
$$-3786a_1a_2a_3a_4a_6^2 + 1296a_3^2a_4a_6^2 - 572a_1^2a_4^2a_6^2 + 2414a_2a_4^2a_6^2$$
$$+3a_1^5a_5a_6^2 + 163a_1^3a_2a_5a_6^2 - 308a_1a_2^2a_5a_6^2 - 592a_1^2a_3a_5a_6^2$$
$$-1068a_2a_3a_5a_6^2 + 1232a_1a_4a_5a_6^2 + 96a_5^2a_6^2 + 10a_1^4a_6^3 - 782a_1^2a_2a_6^3$$
$$+2040a_2^2a_6^3 + 1296a_1a_3a_6^3 - 2592a_4a_6^3$$

$$c_{12} = 2a_1a_2a_3^3a_4^3 + 2a_3^4a_4^3 + 2a_1^3a_2a_3a_4^4 - 7a_1a_2^2a_3a_4^4 + 6a_1^2a_3^2a_4^4$$
$$-14a_2a_3^2a_4^4 + 2a_1^4a_4^5 - 14a_1^2a_2a_4^5 + 22a_2^2a_4^5 - 10a_1a_3a_4^5$$
$$+9a_4^6 + 2a_1^3a_3^4a_4a_5 - 5a_1a_2a_3^4a_4a_5 - 9a_3^5a_4a_5 + 2a_1^5a_3^2a_4^2a_5$$
$$-8a_1^3a_2a_3^2a_4^2a_5 + 6a_1a_2^2a_3^2a_4^2a_5 - 37a_1^2a_3^3a_4^2a_5 + 62a_2a_3^3a_4^2a_5$$

$$+4a_1^5a_2a_4^3a_5 - 28a_1^3a_2^2a_4^3a_5 + 48a_1a_2^3a_4^3a_5 - 16a_1^4a_3a_4^3a_5$$
$$+74a_1^2a_2a_3a_4^3a_5 - 88a_2^2a_3a_4^3a_5 + 56a_1a_3^2a_4^3a_5 - 7a_1^3a_4^4a_5$$
$$+50a_1a_2a_4^4a_5 - 51a_3a_4^4a_5 + 3a_1^5a_3^3a_5^2 - 20a_1^3a_2a_3^3a_5^2 + 32a_1a_2^2a_3^3a_5^2$$
$$+9a_2a_3^4a_5^2 + 4a_1^7a_3a_4a_5^2 - 38a_1^5a_2a_3a_4a_5^2 + 122a_1^3a_2^2a_3a_4a_5^2$$
$$-140a_1a_2^3a_3a_4a_5^2 - 32a_1^4a_3^2a_4a_5^2 + 172a_1^2a_2a_3^2a_4a_5^2 - 112a_2^2a_3^2a_4a_5^2$$
$$-4a_1a_3^3a_4a_5^2 - 17a_1^6a_4^2a_5^2 + 126a_1^4a_2a_4^2a_5^2 - 256a_1^2a_2^2a_4^2a_5^2$$
$$+108a_2^3a_4^2a_5^2 + 111a_1^3a_3a_4^2a_5^2 - 412a_1a_2a_3a_4^2a_5^2 + 55a_3^2a_4^2a_5^2$$
$$-24a_1^2a_4^3a_5^2 - 104a_2a_4^3a_5^2 + a_1^9a_5^3 - 12a_1^7a_2a_5^3 + 54a_1^5a_2^2a_5^3$$
$$-106a_1^3a_2^3a_5^3 + 72a_1a_2^4a_5^3 - 9a_1^6a_3a_5^3 + 66a_1^4a_2a_3a_5^3 - 136a_1^2a_2^2a_3a_5^3$$
$$+144a_2^3a_3a_5^3 - 19a_1^3a_3^2a_5^3 - 154a_1a_2a_3^2a_5^3 - 90a_3^3a_5^3 + 80a_1^5a_4a_5^3$$
$$-554a_1^3a_2a_4a_5^3 + 878a_1a_2^2a_4a_5^3 - 68a_1^2a_3a_4a_5^3 + 872a_2a_3a_4a_5^3$$
$$+78a_1a_4^2a_5^3 - 116a_1^4a_5^4 + 856a_1^2a_2a_5^4 - 1440a_2^2a_5^4 - 157a_1a_3a_5^4$$
$$-94a_4a_5^4 + 2a_1^3a_3^5a_6 - 9a_1a_2a_3^5a_6 + 27a_3^6a_6 + 8a_1^5a_3^3a_4a_6$$
$$-52a_1^3a_2a_3^3a_4a_6 + 68a_1a_2^2a_3^3a_4a_6 + 111a_1^2a_3^4a_4a_6 - 198a_2a_3^4a_4a_6$$
$$+4a_1^7a_3a_4^2a_6 - 44a_1^5a_2a_3a_4^2a_6 + 144a_1^3a_2^2a_3a_4^2a_6 - 136a_1a_2^3a_3a_4^2a_6$$
$$+56a_1^4a_3^2a_4^2a_6 - 334a_1^2a_2a_3^2a_4^2a_6 + 460a_2^2a_3^2a_4^2a_6 - 162a_1a_3^3a_4^2a_6$$
$$+8a_1^6a_4^3a_6 - 92a_1^4a_2a_4^3a_6 + 336a_1^2a_2^2a_4^3a_6 - 368a_2^3a_4^3a_6 - 18a_1^3a_3a_4^3a_6$$
$$+40a_1a_2a_3a_4^3a_6 + 56a_3^2a_4^3a_6 - 72a_1^2a_4^4a_6 + 320a_2a_4^4a_6 + 8a_1^7a_3^2a_5a_6$$
$$-70a_1^5a_2a_3^2a_5a_6 + 166a_1^3a_2^2a_3^2a_5a_6 - 64a_1a_2^3a_3^2a_5a_6 + 106a_1^4a_3^3a_5a_6$$
$$-376a_1^2a_2a_3^3a_5a_6 - 9a_1a_3^4a_5a_6 + 6a_1^9a_4a_5a_6 - 72a_1^7a_2a_4a_5a_6$$
$$+304a_1^5a_2^2a_4a_5a_6 - 528a_1^3a_2^3a_4a_5a_6 + 320a_1a_2^4a_4a_5a_6$$
$$+42a_1^6a_3a_4a_5a_6 - 152a_1^4a_2a_3a_4a_5a_6 - 88a_1^2a_2^2a_3a_4a_5a_6$$
$$+224a_2^3a_3a_4a_5a_6 - 432a_1^3a_3^2a_4a_5a_6 + 1784a_1a_2a_3^2a_4a_5a_6$$
$$+234a_3^3a_4a_5a_6 - 90a_1^5a_4^2a_5a_6 + 508a_1^3a_2a_4^2a_5a_6 - 680a_1a_2^2a_4^2a_5a_6$$
$$+454a_1^2a_3a_4^2a_5a_6 - 2264a_2a_3a_4^2a_5a_6 + 148a_1a_4^3a_5a_6 - 22a_1^8a_5^2a_6$$
$$+256a_1^6a_2a_5^2a_6 - 1058a_1^4a_2^2a_5^2a_6 + 1792a_1^2a_2^3a_5^2a_6 - 1008a_2^4a_5^2a_6$$
$$-178a_1^5a_3a_5^2a_6 + 998a_1^3a_2a_3a_5^2a_6 - 1430a_1a_2^2a_3a_5^2a_6 + 592a_1^2a_3^2a_5^2a_6$$
$$-612a_2a_3^2a_5^2a_6 + 322a_1^4a_4a_5^2a_6 - 1764a_1^2a_2a_4a_5^2a_6 + 2764a_2^2a_4a_5^2a_6$$

$$-1244a_1a_3a_4a_5^2a_6 + 284a_4^2a_5^2a_6 - 152a_1^3a_5^3a_6 + 144a_1a_2a_5^3a_6$$
$$+1224a_3a_5^3a_6 + 3a_1^9a_3a_6^2 - 32a_1^7a_2a_3a_6^2 + 112a_1^5a_2^2a_3a_6^2$$
$$-96a_1^3a_2^3a_3a_6^2 - 112a_1a_2^4a_3a_6^2 + 28a_1^6a_3^2a_6^2 - 278a_1^4a_2a_3^2a_6^2$$
$$+664a_1^2a_2^2a_3^2a_6^2 - 144a_2^3a_3^2a_6^2 - 52a_1^3a_3^3a_6^2 + 234a_1a_2a_3^3a_6^2 - 324a_3^4a_6^2$$
$$-22a_1^8a_4a_6^2 + 224a_1^6a_2a_4a_6^2 - 688a_1^4a_2^2a_4a_6^2 + 416a_1^2a_2^3a_4a_6^2$$
$$+608a_2^4a_4a_6^2 - 338a_1^5a_3a_4a_6^2 + 2308a_1^3a_2a_3a_4a_6^2 - 3584a_1a_2^2a_3a_4a_6^2$$
$$-942a_1^2a_3^2a_4a_6^2 + 1260a_2a_3^2a_4a_6^2 + 228a_1^4a_4^2a_6^2 - 1220a_1^2a_2a_4^2a_6^2$$
$$+1040a_2^2a_4^2a_6^2 + 1782a_1a_3a_4^2a_6^2 - 1012a_4^3a_6^2 + 26a_1^7a_5a_6^2$$
$$-318a_1^5a_2a_5a_6^2 + 1160a_1^3a_2^2a_5a_6^2 - 1152a_1a_2^3a_5a_6^2 + 560a_1^4a_3a_5a_6^2$$
$$-2996a_1^2a_2a_3a_5a_6^2 + 2592a_2^2a_3a_5a_6^2 - 252a_1a_3^2a_5a_6^2 - 830a_1^3a_4a_5a_6^2$$
$$+3384a_1a_2a_4a_5a_6^2 - 792a_3a_4a_5a_6^2 + 944a_1^2a_5^2a_6^2 - 2880a_2a_5^2a_6^2$$
$$+170a_1^6a_6^3 - 1216a_1^4a_2a_6^3 + 2384a_1^2a_2^2a_6^3$$

$$c_{13} = a_2^2a_3^2a_4^4 + 2a_1a_3^3a_4^4 + a_1^2a_2^2a_4^5 - 4a_2^3a_4^5 + 2a_1^3a_3a_4^5 - 6a_1a_2a_3a_4^5$$
$$-2a_3^2a_4^5 - 2a_1^2a_4^6 + a_2a_4^6 + 3a_1^2a_2a_3^3a_4^2a_5 - 6a_2^2a_3^3a_4^2a_5$$
$$-9a_1a_3^4a_4^2a_5 + 3a_1^4a_2a_3a_4^3a_5 - 18a_1^2a_2^2a_3a_4^3a_5 + 26a_2^3a_3a_4^3a_5$$
$$-6a_1^3a_3^2a_4^3a_5 + 7a_1a_2a_3^2a_4^3a_5 + 9a_3^3a_4^3a_5 + 3a_1^5a_4^4a_5 - 30a_1^3a_2a_4^4a_5$$
$$+69a_1a_2^2a_4^4a_5 + 17a_2a_3a_4^4a_5 + 23a_1a_4^5a_5 + a_1^4a_3^4a_5^2 - 6a_1^2a_2a_3^4a_5^2$$
$$+9a_2^2a_3^4a_5^2 + a_1^6a_3^2a_4a_5^2 - 9a_1^4a_2a_3^2a_4a_5^2 + 30a_1^2a_2^2a_3^2a_4a_5^2$$
$$-42a_2^3a_3^2a_4a_5^2 - 25a_1^3a_3^3a_4a_5^2 + 93a_1a_2a_3^3a_4a_5^2 + 3a_1^6a_2a_4^2a_5^2$$
$$-23a_1^4a_2^2a_4^2a_5^2 + 48a_1^2a_2^3a_4^2a_5^2 - 18a_2^4a_4^2a_5^2 - 23a_1^5a_3a_4^2a_5^2$$
$$+179a_1^3a_2a_3a_4^2a_5^2 - 333a_1a_2^2a_3a_4^2a_5^2 + 75a_1^2a_3^2a_4^2a_5^2 - 84a_2a_3^2a_4^2a_5^2$$
$$+3a_1^4a_4^3a_5^2 + 34a_1^2a_2a_4^3a_5^2 - 164a_2^2a_4^3a_5^2 - 182a_1a_3a_4^3a_5^2 - 5a_4^4a_5^2$$
$$+a_1^8a_3a_5^3 - 11a_1^6a_2a_3a_5^3 + 47a_1^4a_2^2a_3a_5^3 - 90a_1^2a_2^3a_3a_5^3 + 54a_2^4a_3a_5^3$$
$$-11a_1^5a_3^2a_5^3 + 61a_1^3a_2a_3^2a_5^3 - 21a_1a_2^2a_3^2a_5^3 - 75a_1^2a_3^3a_5^3 - 99a_2a_3^3a_5^3$$
$$-13a_1^7a_4a_5^3 + 116a_1^5a_2a_4a_5^3 - 330a_1^3a_2^2a_4a_5^3 + 306a_1a_2^3a_4a_5^3$$
$$+40a_1^4a_3a_4a_5^3 - 483a_1^2a_2a_3a_4a_5^3 + 858a_2^2a_3a_4a_5^3 + 285a_1a_3^2a_4a_5^3$$
$$-37a_1^3a_4^2a_5^3 + 252a_1a_2a_4^2a_5^3 + 117a_3a_4^2a_5^3 + 29a_1^6a_5^4 - 278a_1^4a_2a_5^4$$

$$+855a_1^2a_2^2a_5^4 - 864a_2^3a_5^4 + 113a_1^3a_3a_5^4 - 219a_1a_2a_3a_5^4 - 234a_3^2a_5^4$$
$$-69a_1^2a_4a_5^4 - 222a_2a_4a_5^4 + 132a_1a_5^5 + 2a_1^4a_3^4a_4a_6 - 9a_1^2a_2a_3^4a_4a_6$$
$$+27a_1a_3^5a_4a_6 + 2a_1^6a_3^2a_4^2a_6 - 15a_1^4a_2a_3^2a_4^2a_6 + 24a_1^2a_2^2a_3^2a_4^2a_6$$
$$+6a_2^3a_3^2a_4^2a_6 + 28a_1^3a_3^3a_4^2a_6 - 81a_1a_2a_3^3a_4^2a_6 - 27a_3^4a_4^2a_6$$
$$-4a_1^4a_2^2a_4^3a_6 + 24a_1^2a_2^3a_4^3a_6 - 32a_2^4a_4^3a_6 - 7a_1^5a_3a_4^3a_6$$
$$+22a_1^3a_2a_3a_4^3a_6 + 10a_1a_2^2a_3a_4^3a_6 - 59a_1^2a_3^2a_4^3a_6 - 18a_2a_3^2a_4^3a_6$$
$$+27a_1^4a_4^4a_6 - 136a_1^2a_2a_4^4a_6 + 174a_2^2a_4^4a_6 + 145a_1a_3a_4^4a_6 - 118a_4^5a_6$$
$$+6a_1^6a_3^3a_5a_6 - 49a_1^4a_2a_3^3a_5a_6 + 102a_1^2a_2^2a_3^3a_5a_6 - 18a_2^3a_3^3a_5a_6$$
$$+81a_1^3a_3^4a_5a_6 - 270a_1a_2a_3^4a_5a_6 + 6a_1^8a_3a_4a_5a_6 - 74a_1^6a_2a_3a_4a_5a_6$$
$$+294a_1^4a_2^2a_3a_4a_5a_6 - 408a_1^2a_2^3a_3a_4a_5a_6 + 120a_2^4a_3a_4a_5a_6$$
$$+117a_1^5a_3^2a_4a_5a_6 - 708a_1^3a_2a_3^2a_4a_5a_6 + 942a_1a_2^2a_3^2a_4a_5a_6$$
$$-27a_1^2a_3^3a_4a_5a_6 + 540a_2a_3^3a_4a_5a_6 + 12a_1^7a_4^2a_5a_6$$
$$-104a_1^5a_2a_4^2a_5a_6 + 328a_1^3a_2^2a_4^2a_5a_6 - 396a_1a_2^3a_4^2a_5a_6$$
$$-139a_1^4a_3a_4^2a_5a_6 + 864a_1^2a_2a_3a_4^2a_5a_6 - 1002a_2^2a_3a_4^2a_5a_6$$
$$-387a_1a_3^2a_4^2a_5a_6 - 92a_1^3a_4^3a_5a_6 + 136a_1a_2a_4^3a_5a_6 + 450a_3a_4^3a_5a_6$$
$$+3a_1^{10}a_5^2a_6 - 40a_1^8a_2a_5^2a_6 + 206a_1^6a_2^2a_5^2a_6 - 506a_1^4a_2^3a_5^2a_6$$
$$+576a_1^2a_2^4a_5^2a_6 - 216a_2^5a_5^2a_6 + 19a_1^7a_3a_5^2a_6 - 136a_1^5a_2a_3a_5^2a_6$$
$$+316a_1^3a_2^2a_3a_5^2a_6 - 318a_1a_2^3a_3a_5^2a_6 - 152a_1^4a_3^2a_5^2a_6 + 999a_1^2a_2a_3^2a_5^2a_6$$
$$-1260a_2^2a_3^2a_5^2a_6 + 270a_1a_3^3a_5^2a_6 - 26a_1^6a_4a_5^2a_6 + 222a_1^4a_2a_4a_5^2a_6$$
$$-924a_1^2a_2^2a_4a_5^2a_6 + 1644a_2^3a_4a_5^2a_6 + 569a_1^3a_3a_4a_5^2a_6$$
$$-2208a_1a_2a_3a_4a_5^2a_6 - 540a_3^2a_4a_5^2a_6 + 192a_1^2a_4^2a_5^2a_6$$
$$-204a_2a_4^2a_5^2a_6 - 2a_1^5a_5^3a_6 + 238a_1^3a_2a_5^3a_6 - 744a_1a_2^2a_5^3a_6$$
$$-1203a_1^2a_3a_5^3a_6 + 3852a_2a_3a_5^3a_6 + 336a_1a_4a_5^3a_6 - 792a_5^4a_6$$
$$+3a_1^8a_3^2a_6^2 - 30a_1^6a_2a_3^2a_6^2 + 102a_1^4a_2^2a_3^2a_6^2 - 535a_1^3a_3a_4^2a_6^2$$
$$+1782a_1a_2a_3a_4^2a_6^2 + 216a_3^2a_4^2a_6^2 + 326a_1^2a_4^3a_6^2 - 900a_2a_4^3a_6^2$$
$$-23a_1^9a_5a_6^2 + 276a_1^7a_2a_5a_6^2 - 1228a_1^5a_2^2a_5a_6^2 + 2368a_1^3a_2^3a_5a_6^2$$
$$-1632a_1a_2^4a_5a_6^2 - 287a_1^6a_3a_5a_6^2 + 2232a_1^4a_2a_3a_5a_6^2$$
$$-5214a_1^2a_2^2a_3a_5a_6^2 + 3528a_2^3a_3a_5a_6^2 - 639a_1^3a_3^2a_5a_6^2$$

$$+1080a_1a_2a_3^2a_5a_6^2 + 402a_1^5a_4a_5a_6^2 - 2542a_1^3a_2a_4a_5a_6^2$$
$$+4008a_1a_2^2a_4a_5a_6^2 + 1566a_1^2a_3a_4a_5a_6^2 - 2160a_2a_3a_4a_5a_6^2$$
$$-1224a_1a_4^2a_5a_6^2 - 793a_1^4a_5^2a_6^2 + 4200a_1^2a_2a_5^2a_6^2 - 5472a_2^2a_5^2a_6^2$$
$$-1080a_1a_3a_5^2a_6^2 + 2160a_4a_5^2a_6^2 - 20a_1^8a_6^3 + 294a_1^6a_2a_6^3$$
$$-1376a_1^4a_2^2a_6^3 + 2472a_1^2a_2^3a_6^3 - 1440a_2^4a_6^3 - 118a_1^5a_3a_6^3$$
$$+774a_1^3a_2a_3a_6^3 - 1080a_1a_2^2a_3a_6^3 + 216a_1^2a_3^2a_6^3 + 444a_1^4a_4a_6^3$$
$$-2412a_1^2a_2a_4a_6^3 + 2160a_2^2a_4a_6^3 + 432a_1a_3a_4a_6^3 - 432a_4^2a_6^3$$
$$+288a_1^3a_5a_6^3 - 432a_1^2a_6^4$$

$$c_{14} = 2a_2a_3^2a_4^5 + 2a_1^2a_2a_4^6 - 8a_2^2a_4^6 + a_1a_3a_4^6 - 4a_4^7 + a_1a_2^2a_3^2a_4^3a_5$$
$$+2a_1^2a_3^3a_4^3a_5 - 15a_2a_3^3a_4^3a_5 + a_1^3a_2^2a_4^4a_5 - 4a_1a_2^3a_4^4a_5 + 2a_1^4a_3a_4^4a_5$$
$$-24a_1^2a_2a_3a_4^4a_5 + 64a_2^2a_3a_4^4a_5 - 16a_1a_3^2a_4^4a_5 - 7a_1^3a_4^5a_5$$
$$+27a_1a_2a_4^5a_5 + 39a_3a_4^5a_5 + a_1^3a_2a_3^3a_4a_5^2 - 3a_1a_2^2a_3^3a_4a_5^2$$
$$-9a_1^2a_3^4a_4a_5^2 + 27a_2a_3^4a_4a_5^2 + a_1^5a_2a_3a_4^2a_5^2 - 8a_1^3a_2^2a_3a_4^2a_5^2$$
$$+15a_1a_2^3a_3a_4^2a_5^2 - 7a_1^4a_3^2a_4^2a_5^2 + 51a_1^2a_2a_3^2a_4^2a_5^2 - 114a_2^2a_3^2a_4^2a_5^2$$
$$+72a_1a_3^3a_4^2a_5^2 + 3a_1^6a_4^3a_5^2 - 32a_1^4a_2a_4^3a_5^2 + 103a_1^2a_2^2a_4^3a_5^2 - 90a_2^3a_4^3a_5^2$$
$$+50a_1^3a_3a_4^3a_5^2 - 166a_1a_2a_3a_4^3a_5^2 - 129a_3^2a_4^3a_5^2 - 2a_1^2a_4^4a_5^2$$
$$-49a_2a_4^4a_5^2 + 2a_1^3a_2^2a_3^2a_5^3 - 9a_1a_2^3a_3^2a_5^3 - 12a_1^4a_3^3a_5^3 + 63a_1^2a_2a_3^3a_5^3$$
$$-27a_2^2a_3^3a_5^3 - 81a_1a_3^4a_5^3 + a_1^7a_2a_4a_5^3 - 7a_1^5a_2^2a_5^3 + 12a_1^3a_2^3a_4a_5^3$$
$$-17a_1^6a_3a_4a_5^3 + 158a_1^4a_2a_3a_4a_5^3 - 426a_1^2a_2^2a_3a_4a_5^3 + 324a_2^3a_3a_4a_5^3$$
$$+67a_1^3a_3^2a_4a_5^3 + 159a_1a_2a_3^2a_4a_5^3 + 135a_3^3a_4a_5^3 + 8a_1^5a_4^2a_5^3$$
$$-64a_1^3a_2a_4^2a_5^3 + 93a_1a_2^2a_4^2a_5^3 - 42a_1^2a_3a_4^2a_5^3 + 336a_2a_3a_4^2a_5^3$$
$$+57a_1a_4^3a_5^3 - 4a_1^8a_5^4 + 43a_1^6a_2a_5^4 - 174a_1^4a_2^2a_5^4 + 324a_1^2a_2^3a_5^4$$
$$-243a_2^4a_5^4 - 7a_1^5a_3a_5^4 - 13a_1^3a_2a_3a_5^4 + 99a_1a_2^2a_3a_5^4 + 153a_1^2a_3^2a_5^4$$
$$-432a_2a_3^2a_5^4 + 11a_1^4a_4a_5^4 + 48a_1^2a_2a_4a_5^4 - 162a_2^2a_4a_5^4 - 210a_1a_3a_4a_5^4$$
$$-60a_4^2a_5^4 - 34a_1^3a_5^5 + 72a_1a_2a_5^5 + 216a_3a_5^5 + 2a_1^3a_2a_3^3a_4^2a_6$$
$$-9a_1a_2^2a_3^3a_4^2a_6 + 27a_2a_3^4a_4^2a_6 + 2a_1^5a_2a_3a_4^3a_6 - 16a_1^3a_2^2a_3a_4^3a_6$$
$$+32a_1a_2^3a_3a_4^3a_6 - 8a_1^4a_3^2a_4^3a_6 + 64a_1^2a_2a_3^2a_4^3a_6 - 138a_2^2a_3^2a_4^3a_6$$

$$-27a_1a_3^3a_4^3a_6 - 6a_1^6a_4^4a_6 + 50a_1^4a_2a_4^4a_6 - 136a_1^2a_2^2a_4^4a_6$$
$$+128a_2^3a_4^4a_6 - 23a_1^3a_3a_4^4a_6 + 69a_1a_2a_3a_4^4a_6 + 27a_3^2a_4^4a_6 + 45a_1^2a_4^5a_6$$
$$-122a_2a_4^5a_6 + 2a_1^5a_3^4a_5a_6 - 15a_1^3a_2a_3^4a_5a_6 + 27a_1a_2^2a_3^4a_5a_6$$
$$+27a_1^2a_3^5a_5a_6 - 81a_2a_3^5a_5a_6 + 2a_1^7a_3^2a_4a_5a_6 - 24a_1^5a_2a_3^2a_4a_5a_6$$
$$+84a_1^3a_2^2a_3^2a_4a_5a_6 - 84a_1a_2^3a_3^2a_4a_5a_6 + 59a_1^4a_3^3a_4a_5a_6$$
$$-342a_1^2a_2a_3^3a_4a_5a_6 + 432a_2^2a_3^3a_4a_5a_6 + 27a_1a_3^4a_4a_5a_6$$
$$-8a_1^5a_2^2a_4^2a_5a_6 + 56a_1^3a_2^3a_4^2a_5a_6 - 96a_1a_2^4a_4^2a_5a_6 + 24a_1^6a_3a_4^2a_5a_6$$
$$-196a_1^4a_2a_3a_4^2a_5a_6 + 474a_1^2a_2^2a_3a_4^2a_5a_6 - 312a_2^3a_3a_4^2a_5a_6$$
$$+15a_1^3a_3^2a_4^2a_5a_6 + 36a_1a_2a_3^2a_4^2a_5a_6 - 81a_3^3a_4^2a_5a_6 + 10a_1^5a_4^3a_5a_6$$
$$-10a_1^3a_2a_4^3a_5a_6 - 108a_1a_2^2a_4^3a_5a_6 - 158a_1^2a_3a_4^3a_5a_6$$
$$+600a_2a_3a_4^3a_5a_6 - 83a_1a_4^4a_5a_6 + 2a_1^9a_3a_5^2a_6 - 25a_1^7a_2a_3a_5^2a_6$$
$$+115a_1^5a_2^2a_3a_5^2a_6 - 1134a_2a_3^2a_4a_5^2a_6 - 232a_1^3a_2^3a_3a_5^2a_6$$
$$+180a_1a_2^4a_3a_5^2a_6 + 29a_1^6a_3^2a_5^2a_6 - 228a_1^4a_2a_3^2a_5^2a_6 + 579a_1^2a_2^2a_3^2a_5^2a_6$$
$$-594a_2^3a_3^2a_5^2a_6 + 24a_1^3a_3^3a_5^2a_6 + 135a_1a_2a_3^3a_5^2a_6 + 81a_3^4a_5^2a_6$$
$$+a_1^8a_4a_5^2a_6 - 16a_1^6a_2a_4a_5^2a_6 + 130a_1^4a_2^2a_4a_5^2a_6 - 492a_1^2a_2^3a_4a_5^2a_6$$
$$+648a_2^4a_4a_5^2a_6 - 90a_1^5a_3a_4a_5^2a_6 + 526a_1^3a_2a_3a_4a_5^2a_6$$
$$-654a_1a_2^2a_3a_4a_5^2a_6 + 9a_1^2a_3^2a_4a_5^2a_6 - 82a_1^4a_4^2a_5^2a_6 + 510a_1^2a_2a_4^2a_5^2a_6$$
$$-606a_2^2a_4^2a_5^2a_6 + 225a_1a_3a_4^2a_5^2a_6 + 138a_4^3a_5^2a_6 + 35a_1^7a_5^3a_6$$
$$-336a_1^5a_2a_5^3a_6 + 1078a_1^3a_2^2a_5^3a_6 - 1152a_1a_2^3a_5^3a_6 + 468a_1^4a_3a_5^3a_6$$
$$-2562a_1^2a_2a_3a_5^3a_6 + 3348a_2^2a_3a_5^3a_6 + 486a_1a_3^2a_5^3a_6 - 302a_1^3a_4a_5^3a_6$$
$$+720a_1a_2a_4a_5^3a_6 - 540a_3a_4a_5^3a_6 + 516a_1^2a_5^4a_6 - 1296a_2a_5^4a_6$$
$$+3a_1^7a_3^3a_6^2 - 29a_1^5a_2a_3^3a_6^2 + 96a_1^3a_2^2a_3^3a_6^2 - 108a_1a_2^3a_3^3a_6^2$$
$$-27a_1^2a_2a_3^4a_6^2 + 81a_2^2a_3^4a_6^2 + 2a_1^9a_3a_4a_6^2 - 32a_1^7a_2a_3a_4a_6^2$$
$$+184a_1^5a_2^2a_3a_4a_6^2 - 448a_1^3a_2^3a_3a_4a_6^2 + 384a_1a_2^4a_3a_4a_6^2 + 5a_1^6a_3^2a_4a_6^2$$
$$-78a_1^4a_2a_3^2a_4a_6^2 + 288a_1^2a_2^2a_3^2a_4a_6^2 - 216a_2^3a_3^2a_4a_6^2 + 27a_1^3a_3^3a_4a_6^2$$
$$-162a_1a_2a_3^3a_4a_6^2 + 10a_1^8a_4^2a_6^2 - 80a_1^6a_2a_4^2a_6^2 + 160a_1^4a_2^2a_4^2a_6^2$$
$$+96a_1^2a_2^3a_4^2a_6^2 - 384a_2^4a_4^2a_6^2 + 54a_1^5a_3a_4^2a_6^2 - 221a_1^3a_2a_3a_4^2a_6^2$$
$$+144a_1a_2^2a_3a_4^2a_6^2 + 81a_1^2a_3^2a_4^2a_6^2 - 54a_2a_3^2a_4^2a_6^2 - 53a_1^4a_4^3a_6^2$$

$$+62a_1^2a_2a_4^3a_6^2 + 336a_2^2a_4^3a_6^2 - 54a_1a_3a_4^3a_6^2 - 27a_4^4a_6^2 + 3a_1^{11}a_5a_6^2$$
$$-44a_1^9a_2a_5a_6^2 + 256a_1^7a_2^2a_5a_6^2 - 736a_1^5a_2^3a_5a_6^2 + 1040a_1^3a_2^4a_5a_6^2$$
$$-576a_1a_2^5a_5a_6^2 + 26a_1^8a_3a_5a_6^2 - 278a_1^6a_2a_3a_5a_6^2 + 1048a_1^4a_2^2a_3a_5a_6^2$$
$$-1608a_1^2a_2^3a_3a_5a_6^2 + 864a_2^4a_3a_5a_6^2 + 31a_1^5a_3^2a_5a_6^2 - 93a_1^3a_2a_3^2a_5a_6^2$$
$$-108a_1a_2^2a_3^2a_5a_6^2 - 297a_1^2a_3^3a_5a_6^2 + 648a_2a_3^3a_5a_6^2 - 115a_1^7a_4a_5a_6^2$$
$$+1048a_1^5a_2a_4a_5a_6^2 - 3080a_1^3a_2^2a_4a_5a_6^2 + 2880a_1a_2^3a_4a_5a_6^2$$
$$-554a_1^4a_3a_4a_5a_6^2 + 2448a_1^2a_2a_3a_4a_5a_6^2 - 1728a_2^2a_3a_4a_5a_6^2$$
$$+108a_1a_3^2a_4a_5a_6^2 + 807a_1^3a_4^2a_5a_6^2 - 2844a_1a_2a_4^2a_5a_6^2 + 324a_3a_4^2a_5a_6^2$$
$$+103a_1^6a_5^2a_6^2 - 1048a_1^4a_2a_5^2a_6^2 + 3318a_1^2a_2^2a_5^2a_6^2 - 3240a_2^3a_5^2a_6^2$$
$$+39a_1^3a_3a_5^2a_6^2 - 540a_1a_2a_3a_5^2a_6^2 - 648a_3^2a_5^2a_6^2 - 810a_1^2a_4a_5^2a_6^2$$
$$+4104a_2a_4a_5^2a_6^2 - 648a_1a_5^3a_6^2 - 8a_1^{10}a_6^3 + 88a_1^8a_2a_6^3 - 352a_1^6a_2^2a_6^3$$
$$+608a_1^4a_2^3a_6^3 - 384a_1^2a_2^4a_6^3 - 29a_1^7a_3a_6^3 + 200a_1^5a_2a_3a_6^3$$
$$-492a_1^3a_2^2a_3a_6^3 + 432a_1a_2^3a_3a_6^3 - 27a_1^4a_3^2a_6^3 + 378a_1^2a_2a_3^2a_6^3$$
$$-648a_2^2a_3^2a_6^3 - 6a_1^6a_4a_6^3 + 228a_1^4a_2a_4a_6^3 - 936a_1^2a_2^2a_4a_6^3$$
$$+864a_2^3a_4a_6^3 - 270a_1^3a_3a_4a_6^3 + 648a_1a_2a_3a_4a_6^3 + 162a_1^2a_4^2a_6^3$$
$$-216a_2a_4^2a_6^3 - 18a_1^5a_5a_6^3 + 72a_1^3a_2a_5a_6^3 + 756a_1^2a_3a_5a_6^3$$
$$-1296a_2a_3a_5a_6^3 - 864a_1a_4a_5a_6^3 + 1296a_5^2a_6^3 + 189a_1^4a_6^4$$
$$-1080a_1^2a_2a_6^4 + 1296a_2^2a_6^4$$

$$c_{15} = a_3^2a_4^6 + a_1^2a_4^7 - 4a_2a_4^7 + a_1a_2a_3^2a_4^4a_5 - 9a_3^3a_4^4a_5 + a_1^3a_2a_4^5a_5$$
$$-4a_1a_2^2a_4^5a_5 - 10a_1^2a_3a_4^5a_5 + 38a_2a_3a_4^5a_5 + 2a_1a_4^6a_5 + a_2^3a_3^2a_4^2a_5^2$$
$$+a_1^3a_3^3a_4^2a_5^2 - 9a_1a_2a_3^3a_4^2a_5^2 - 27a_3^4a_4^2a_5^2 + a_1^2a_2^3a_4^3a_5^2 - 4a_2^4a_4^3a_5^2$$
$$+a_1^5a_3a_4^3a_5^2 - 14a_1^3a_2a_3a_4^3a_5^2 + 38a_1a_2^2a_3a_4^3a_5^2 + 33a_1^2a_3^2a_4^3a_5^2$$
$$-117a_2a_3^2a_4^3a_5^2 + 2a_1^4a_4^4a_5^2 + 18a_1^2a_2a_4^4a_5^2 - 31a_2^2a_4^4a_5^2$$
$$-16a_1a_3a_4^4a_5^2 - 3a_4^5a_5^2 + a_1^2a_2^2a_3^3a_5^3 - 4a_2^3a_3^3a_5^3 - 4a_1^3a_3^4a_5^3$$
$$+18a_1a_2a_3^4a_5^3 - 27a_3^5a_5^3 + a_1^4a_2^2a_3a_4a_5^3 - 9a_1^2a_2^3a_3a_4a_5^3$$
$$+18a_2^4a_3a_4a_5^3 - 4a_1^5a_3^2a_4a_5^3 + 38a_1^3a_2a_3^2a_4a_5^3 - 80a_1a_2^2a_3^2a_4a_5^3$$
$$-33a_1^2a_3^3a_4a_5^3 + 117a_2a_3^3a_4a_5^3 + a_1^7a_4^2a_5^3 - 10a_1^5a_2a_4^2a_5^3$$
$$+33a_1^3a_2^2a_4^2a_5^3 - 33a_1a_2^3a_4^2a_5^3 - 18a_1^4a_3a_4^2a_5^3 - 120a_1^2a_2a_3a_4^2a_5^3$$

$$+157a_2^2a_3a_4^2a_5^3 + 33a_1a_3^2a_4^2a_5^3 + 12a_1^3a_4^3a_5^3 + 41a_1a_2a_4^3a_5^3$$
$$+22a_3a_4^3a_5^3 + a_1^6a_2^2a_5^4 - 9a_1^4a_2^3a_5^4 + 27a_1^2a_2^4a_5^4 - 27a_2^5a_5^4$$
$$-4a_1^7a_3a_5^4 + 38a_1^5a_2a_3a_5^4 - 117a_1^3a_2^2a_3a_5^4 - 117a_1a_2^3a_3a_5^4$$
$$-31a_1^4a_3^2a_5^4 + 157a_1^2a_2a_3^2a_5^4 - 186a_2^2a_3^2a_5^4 + 18a_1a_3^3a_5^4 + 2a_1^6a_4a_5^4$$
$$-16a_1^4a_2a_4a_5^4 + 33a_1^2a_2^2a_4a_5^4 - 18a_2^3a_4a_5^4 - 41a_1^3a_3a_4a_5^4$$
$$-86a_1a_2a_3a_4a_5^4 - 36a_3^2a_4a_5^4 + 6a_1^2a_4^2a_5^4 - 68a_2a_4^2a_5^4 - 3a_1^5a_5^5$$
$$+22a_1^3a_2a_5^5 - 36a_1a_2^2a_5^5 - 68a_1^2a_3a_5^5 + 168a_2a_3a_5^5 + 40a_1a_4a_5^5$$
$$+32a_5^6 + a_1^2a_2^2a_3^2a_4^3a_6 - 4a_2^3a_3^2a_4^3a_6 - 2a_1^3a_3^3a_4^3a_6 + 9a_1a_2a_3^3a_4^3a_6$$
$$+a_1^4a_2^2a_4^4a_6 - 8a_1^2a_2^3a_4^4a_6 + 16a_2^4a_4^4a_6 - 2a_1^5a_3a_4^4a_6$$
$$+17a_1^3a_2a_3a_4^4a_6 - 36a_1a_2^2a_3a_4^4a_6 - 3a_1^2a_3^2a_4^4a_6 - 4a_1^4a_4^5a_6$$
$$-17a_1^2a_2a_4^5a_6 - 4a_2^2a_4^5a_6 - 3a_1a_3a_4^5a_6 + a_1^4a_2a_3^3a_4a_5a_6$$
$$+9a_1^2a_2^2a_3^3a_4a_5a_6 + 18a_2^3a_3^3a_4a_5a_6 + 9a_1^3a_3^4a_4a_5a_6$$
$$-27a_1a_2a_3^4a_4a_5a_6 + a_1^6a_2a_3a_4^2a_5a_6 - 14a_1^4a_2^2a_3a_4^2a_5a_6$$
$$+60a_1^2a_2^3a_3a_4^2a_5a_6 - 80a_2^4a_3a_4^2a_5a_6 + 8a_1^5a_3^2a_4^2a_5a_6$$
$$-63a_1^3a_2a_3^2a_4^2a_5a_6 + 117a_1a_2^2a_3^2a_4^2a_5a_6 - 27a_2a_3^3a_4^2a_5a_6$$
$$-3a_1^7a_4^3a_5a_6 + 24a_1^5a_2a_4^3a_5a_6 - 54a_1^3a_2^2a_4^3a_5a_6 + 24a_1a_2^3a_4^3a_5a_6$$
$$+6a_1^4a_3a_4^3a_5a_6 - 54a_1^2a_2a_3a_4^3a_5a_6 + 138a_2^2a_3a_4^3a_5a_6$$
$$+54a_1a_3^2a_4^3a_5a_6 + 28a_1^3a_4^4a_5a_6 - 137a_1a_2a_4^4a_5a_6 + 9a_3a_4^4a_5a_6$$
$$-a_1^4a_2^2a_3^2a_5^2a_6 + 6a_1^2a_2^3a_3^2a_5^2a_6 - 6a_2^4a_3^2a_5^2a_6 + 3a_1^3a_2a_3^3a_5^2a_6$$
$$-27a_1a_2^2a_3^3a_5^2a_6 + 81a_2a_3^4a_5^2a_6 + a_1^8a_2a_4a_5^2a_6 - 15a_1^6a_2^2a_4a_5^2a_6$$
$$+80a_1^4a_2^3a_4a_5^2a_6 - 180a_1^2a_2^4a_4a_5^2a_6 + 144a_2^5a_4a_5^2a_6 + 9a_1^7a_3a_4a_5^2a_6$$
$$-64a_1^5a_2a_3a_4a_5^2a_6 + 119a_1^3a_2^2a_3a_4a_5^2a_6 - 26a_1a_2^3a_3a_4a_5^2a_6$$
$$+3a_1^4a_3^2a_4a_5^2a_6 + 135a_1^2a_2a_3^2a_4a_5^2a_6 - 378a_2^2a_3^2a_4a_5^2a_6$$
$$-108a_1a_3^3a_4a_5^2a_6 - 38a_1^4a_2a_4^2a_5^2a_6 + 210a_1^2a_2^2a_4^2a_5^2a_6$$
$$-286a_2^3a_4^2a_5^2a_6 - 94a_1^3a_3a_4^2a_5^2a_6 + 261a_1a_2a_3a_4^2a_5^2a_6 - 108a_3^2a_4^2a_5^2a_6$$
$$+20a_1^2a_4^3a_5^2a_6 + 210a_2a_4^3a_5^2a_6 - 4a_1^9a_5^3a_6 + 49a_1^7a_2a_5^3a_6$$
$$-227a_1^5a_2^2a_5^3a_6 + 468a_1^3a_2^3a_5^3a_6 - 360a_1a_2^4a_5^3a_6 - 35a_1^6a_3a_5^3a_6$$
$$+313a_1^4a_2a_3a_5^3a_6 - 905a_1^2a_2^2a_3a_5^3a_6 + 840a_2^3a_3a_5^3a_6 - 11a_1^3a_3^2a_5^3a_6$$

$$
\begin{aligned}
&+18a_1a_2a_3^2a_5^3a_6+216a_3^3a_5^3a_6+49a_1^5a_4a_5^3a_6-282a_1^3a_2a_4a_5^3a_6\\
&+416a_1a_2^2a_4a_5^3a_6+126a_1^2a_3a_4a_5^3a_6-468a_2a_3a_4a_5^3a_6\\
&-192a_1a_4^2a_5^3a_6-54a_1^4a_5^4a_6+308a_1^2a_2a_5^4a_6-456a_2^2a_5^4a_6\\
&+72a_1a_3a_5^4a_6+144a_4a_5^4a_6+a_1^6a_3^4a_6^2-9a_1^4a_2a_3^4a_6^2+27a_1^2a_2^2a_3^4a_6^2\\
&-27a_2^3a_3^4a_6^2+a_1^8a_3^2a_4a_6^2-15a_1^6a_2a_3^2a_4a_6^2+80a_1^4a_2^2a_3^2a_4a_6^2\\
&-180a_1^2a_2^3a_3^2a_4a_6^2+144a_2^4a_3^2a_4a_6^2+3a_1^5a_3^3a_4a_6^2-27a_1^3a_2a_3^3a_4a_6^2\\
&+54a_1a_2^2a_3^3a_4a_6^2-2a_1^8a_2a_4^2a_6^2+24a_1^6a_2^2a_4^2a_6^2-104a_1^4a_2^3a_4^2a_6^2\\
&+192a_1^2a_2^4a_4^2a_6^2-128a_2^5a_4^2a_6^2+5a_1^7a_3a_4^2a_6^2-58a_1^5a_2a_3a_4^2a_6^2\\
&+212a_1^3a_2^2a_3a_4^2a_6^2-240a_1a_2^3a_3a_4^2a_6^2+12a_1^4a_3^2a_4^2a_6^2-27a_1^2a_2a_3^2a_4^2a_6^2\\
&-54a_2^2a_3^2a_4^2a_6^2+8a_1^6a_4^3a_6^2-31a_1^4a_2a_4^3a_6^2-60a_1^2a_2^2a_4^3a_6^2\\
&+224a_2^3a_4^3a_6^2+4a_1^3a_3a_4^3a_6^2+18a_1a_2a_3a_4^3a_6^2+9a_1^2a_4^4a_6^2\\
&-27a_2a_4^4a_6^2+a_1^{10}a_3a_5a_6^2-14a_1^8a_2a_3a_5a_6^2+80a_1^6a_2^2a_3a_5a_6^2\\
&-232a_1^4a_2^3a_3a_5a_6^2+336a_1^2a_2^4a_3a_5a_6^2-192a_2^5a_3a_5a_6^2+a_1^7a_3^2a_5a_6^2\\
&-13a_1^5a_2a_3^2a_5a_6^2+60a_1^3a_2^2a_3^2a_5a_6^2-72a_1a_2^3a_3^2a_5a_6^3-27a_1^2a_2a_3^3a_5a_6^2\\
&+10a_1^9a_4a_5a_6^2-122a_1^7a_2a_4a_5a_6^2+544a_1^5a_2^2a_4a_5a_6^2\\
&-1048a_1^3a_2^3a_4a_5a_6^2+736a_1a_2^4a_4a_5a_6^2+31a_1^6a_3a_4a_5a_6^2\\
&-209a_1^4a_2a_3a_4a_5a_6^2+342a_1^2a_2^2a_3a_4a_5a_6^2-72a_2^3a_3a_4a_5a_6^2\\
&-54a_1^3a_3^2a_4a_5a_6^2+324a_1a_2a_3^2a_4a_5a_6^2-93a_1^5a_4^2a_5a_6^2\\
&+695a_1^3a_2a_4^2a_5a_6^2-1188a_1a_2^2a_4^2a_5a_6^2-162a_1^2a_3a_4^2a_5a_6^2\\
&+108a_2a_3a_4^2a_5a_6^2+3a_1^8a_5^2a_6^2+8a_1^6a_2a_5^2a_6^2-192a_1^4a_2^2a_5^2a_6^2\\
&+622a_1^2a_2^3a_5^2a_6^2-600a_2^4a_5^2a_6^2+8a_1^5a_3a_5^2a_6^2-45a_1^3a_2a_3a_5^2a_6^2\\
&+108a_1a_2^2a_3a_5^2a_6^2+108a_1^2a_3^2a_5^2a_6^2-648a_2a_3^2a_5^2a_6^2+35a_1^4a_4a_5^2a_6^2\\
&-666a_1^2a_2a_4a_5^2a_6^2+1512a_2^2a_4a_5^2a_6^2+432a_1a_3a_4a_5^2a_6^2+136a_1^3a_5^3a_6^2\\
&-360a_1a_2a_5^3a_6^2-432a_3a_5^3a_6^2+a_1^{12}a_6^3-16a_1^{10}a_2a_6^3+104a_1^8a_2^2a_6^3\\
&-352a_1^6a_2^3a_6^3+656a_1^4a_2^4a_6^3-640a_1^2a_2^5a_6^3+256a_2^6a_6^3+2a_1^9a_3a_6^3.
\end{aligned}
$$

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